Continuous Random Variables

CSE 312 26Su

Lecture 13

Fun fact!

You play a game where a fair coin is tossed until it comes up heads. The payoff is 2^n dollars, where n is the number of tosses. The expected value of this game is infinite, which is surprising because no one would realistically pay an extremely high entry fee to play. (convince yourself of this using the geometric distribution PMF and LOTUS!)

Announcements

- **HW4 released** and due next Wednesday

What have we done over the past 4 weeks?

Counting

Combinations, permutations, indistinguishable elements, starts and bars, inclusion-exclusion…

Probability foundations

Events, sample space, axioms of probability, expectation, variance

Conditional probability

Conditioning, independence, Bayes’ Rule

Refined our intuition

Especially around Bayes’ Rule

What’s next?

Continuous random variables.

So far our sample spaces have been countable. What happens if we want to choose a random real number?

How do expectation, variance, conditioning, etc. change in this new context?

Mostly analogous to discrete cases, but with integrals instead of sums.

Analysis when it’s inconvenient (or impossible) to exactly calculate probabilities.

Central Limit Theorem (approximating discrete distributions with continuous ones)

Tail Bounds/Concentration (arguing it’s unlikely that a random variable is far from its expectation)

A first taste of making predictions from data (i.e., a bit of ML)

Where Are We?

Last time:

- Zoo of Discrete RVs

Today:

- Continuous Random Variables
  - Probability Density Function
  - Cumulative Distribution Function
  - Expectation and Variance
- Comparing Discrete vs. Continuous
- Continuous Uniform Distribution

Continuous Random Variables

Goal for today is to get intuition on what's different in the continuous case

ASK QUESTIONS (always! but today especially ☺)

Discrete Random Variables

The kind that we've been working with up till now!

The support has *finite* or *countably infinite values*

e.g., number of successes, number of trials till success, attendance at a class are all discrete because they take on a set of finite or countably infinite values

Some random experiments have <u>uncountably-infinite sample spaces</u>

> *How long until the next bus shows up?*

> *Throwing a dart on a board (what location does the dart land?)*

Continuous Random Variables

Random variables with a support of *uncountably-infinite* **values**

> *e.g., RVs that take on any* ***real*** *number in some interval(s) like distance, height, time, etc.*

Why Need New Rules?

Random Experiment: choose a random <u>*real*</u> number between 0 and 1

What's the probability the number is between 0.4 *and* 0.5*?*

> For discrete spaces, we'd ask for $\frac{|E|}{|\Omega|}$

> So we get $\frac{\infty}{\infty}$ ☹

When working with continuous random experiments, we'll almost always use continuous random variables to interact with these spaces

X is the number we choose. We want $\mathbb{P}(0.4 \leq X \leq 0.5)$

Probability Density Function (PDF)

Analogous to PMF in a discrete random variable

How to describe probability of a *single* value?

For **discrete random variables**, we defined the PMF: $p_Y(k) = \mathbb{P}(Y = k)$.

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Can we use the same for continuous random variables?

What is the probability that a person's height is <u>exactly</u> 5.678123589 feet

What is the probability that a bus arrives <u>exactly</u> 6 hours, 23 minutes and 2976427909 seconds from now?

$$p_X(0.135) = \mathbb{P}(X = 0.135) =$$

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$p_X(0.135) = \mathbb{P}(X = 0.135) = \frac{1}{\infty} = 0$ ***No!***

Instead, we use the probability density function (PDF)

$\boldsymbol{f_X(k)}$ is the density (not probability!) of the continuous random variable X at the value k

Probability *Density* Function

For *discrete* **random variables**, we defined the PMF: $p_Y(k) = \mathbb{P}(Y = k)$.

For *continuous* random variables, we use the probability density function

$\boldsymbol{f_X(k)}$ is the density (not probability!) of the continuous random variable X at the value k

Discrete (PMF)	Continuous (PDF)
$p_Y(k) \geq 0$	$f_X(k) \geq 0$
$\sum_{k \in \Omega_Y} p_Y(k) = 1$	$\int_{-\infty}^{\infty} f_X(k)\ \mathrm{d}k = 1$

Probability *Density* Function

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For *continuous* random variables, we use the probability density function

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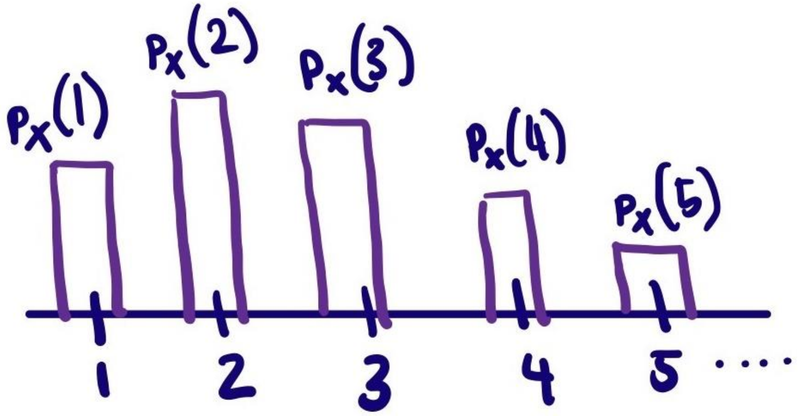
*e.g., PMF for a discrete RV, **X***



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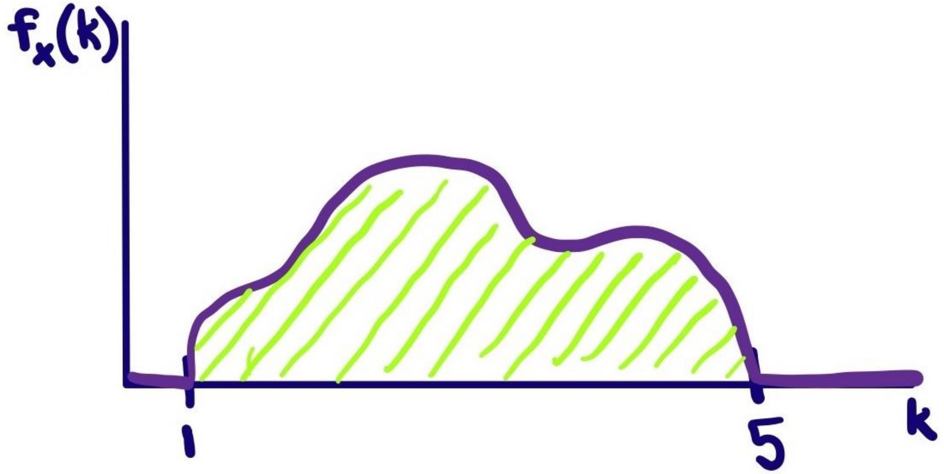


Probability *Density* Function

For *discrete* random variables, we defined the PMF: $p_Y(k) = \mathbb{P}(Y = k)$.

For *continuous* random variables, we use the probability density function

$\boldsymbol{f_X(k)}$ is the density of the continuous random variable X at the value k

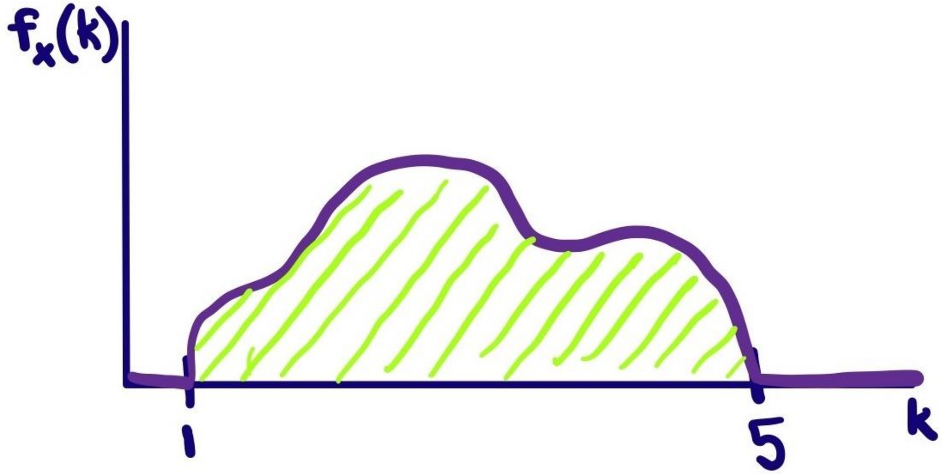
How do we use the PDF?

To compute probabilities of events!

$\mathbb{P}(a \le X \le b) = \int_a^b f_X(z)\,\mathrm{d}z$

integrating is analogous to summing

e.g., PDF for a continuous RV, $\boldsymbol{X}$



Probability *<u>Density</u>* Function (*example)*

X is a **uniform real number** in $[0,1]$. Let's derive the PDF for X!

Idea: Make it "work right" for **events** since single outcomes don't make sense.

$\mathbb{P}(0 \leq X \leq 1) = \qquad = \int$

$\mathbb{P}(X \text{ is negative}) = \qquad = \int$

$\mathbb{P}(.4 \leq X \leq .5) = \qquad = \int$

Probability *Density* Function (*example*)

X is a **uniform real number** in $[0,1]$. Let's derive the PDF for X!

Idea: Make it "work right" for **events** since single outcomes don't make sense.

$$\mathbb{P}(0 \le X \le 1) = \ 1 \ = \int_0^1 f_X(z)\,\mathrm{dz}$$

$$\mathbb{P}(X \text{ is negative}) = 0 = \int_{-\infty}^0 f_X(z)\,\mathrm{d}z$$

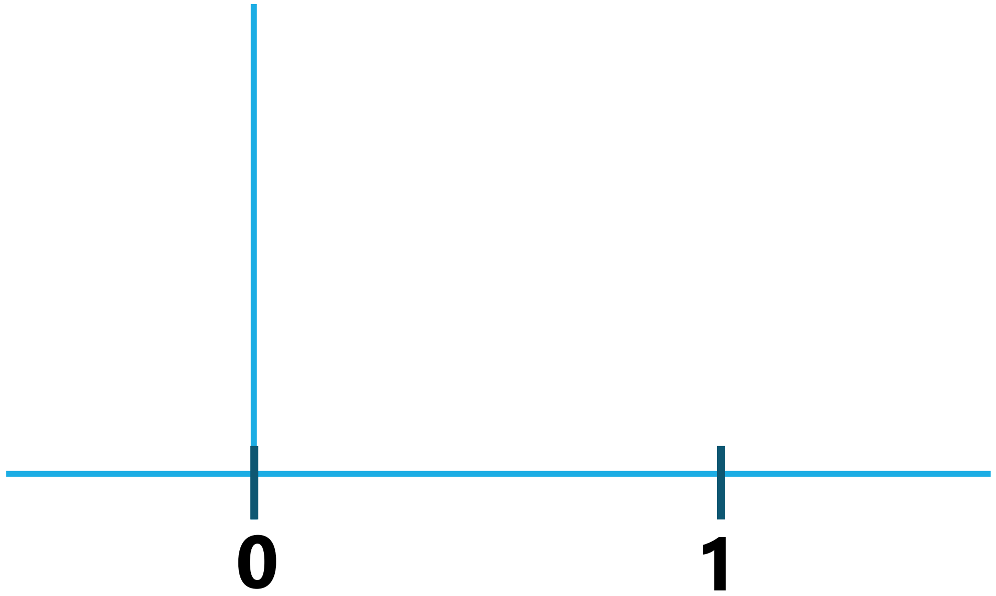
$$\mathbb{P}(.4 \le X \le .5) = 0.1 = \int_{.4}^{.5} f_X(z)\ \mathrm{dz}$$

Probability *Density* Function (*example*)

X is a **uniform real number** in $[0,1]$. Let's derive the PDF for X!

What should $f_X(k)$ be to make all those events integrate to the right values?

$$f_X(k) = \begin{cases} 0 & \text{if } k < 0 \text{ or } k > 1 \\ 1 & \text{if } 0 \le k \le 1 \end{cases}$$

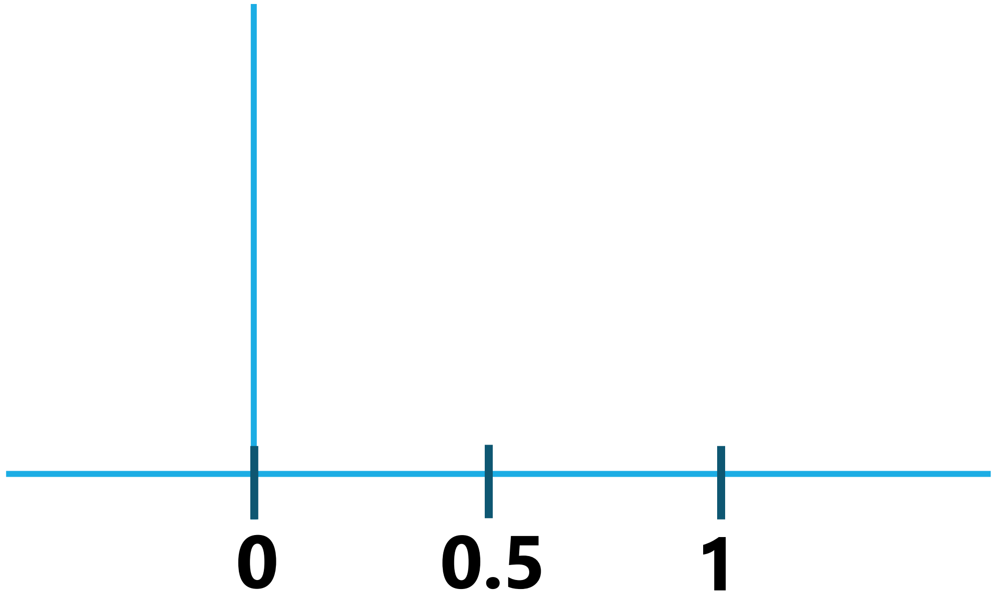


Probability *Density* Function (*example 2*)

X is a **uniform real number** in $[0,1]$. Let's derive the PDF for X!

What should $f_X(k)$ be to make all those events integrate to the right values?

$$f_X(k) = \begin{cases} 0 & \text{if } k < 0 \text{ or } k > 0.5 \\ 2 & \text{if } 0 \leq k \leq 0.5 \end{cases}$$



⚠ **Density ≠ Probability**

$f_X(k) \gg 1$ is possible!

Probability vs. Density

Key idea: integrating the PDF gives us probabilities

But the PDF itself does not give us probabilities

The number that best represents $\mathbb{P}(X = .1)$ is 0

But, this is different from $f_X(.1) = 1$

For continuous probability spaces:

> *Impossible events have probability* 0

> *But even though probability of a specific value is* 0*, it's still possible*

What exactly do the values in the PDF mean?

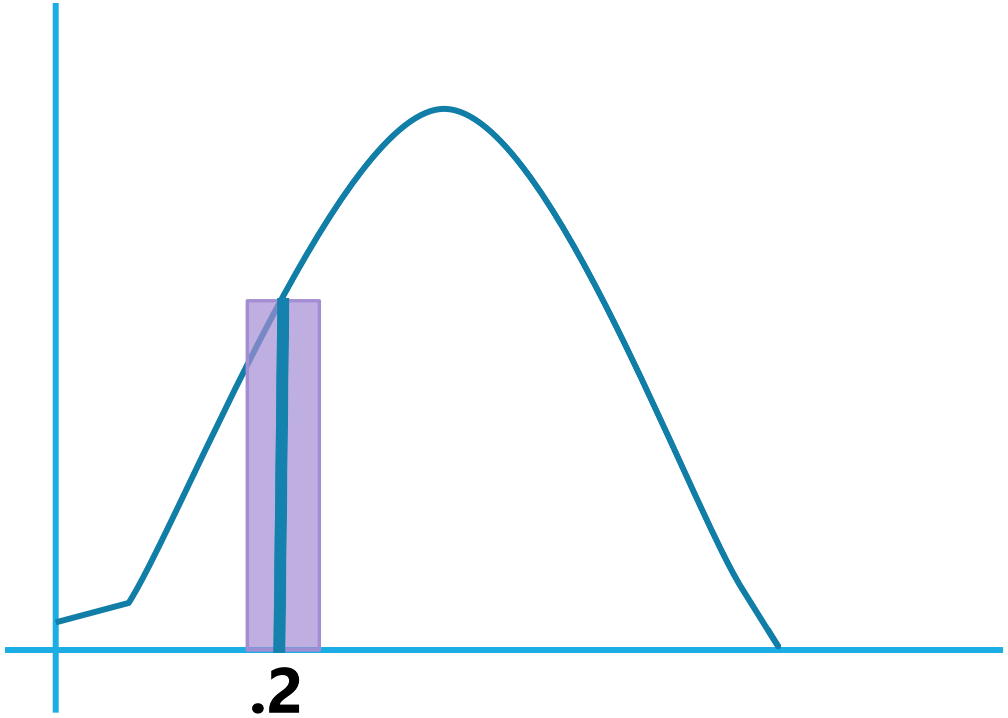
Let's look at the event $X \approx .2$

For a *very* small value of ϵ,

$$\mathbb{P}(X \approx .2) = \ \mathbb{P}(.2 - \epsilon/2 \leq X \leq .2 + \epsilon/2)$$

$$= \int_{.2-\epsilon/2}^{-2+\epsilon/2} f_X(z)\,\mathrm{dz} \approx \text{height} \cdot \text{width}$$

$$= f_X\ (.2) \cdot \epsilon$$



What happens if we look at the ratio

$$\frac{\mathbb{P}(X\approx .2)}{\mathbb{P}(X\approx .5)}$$

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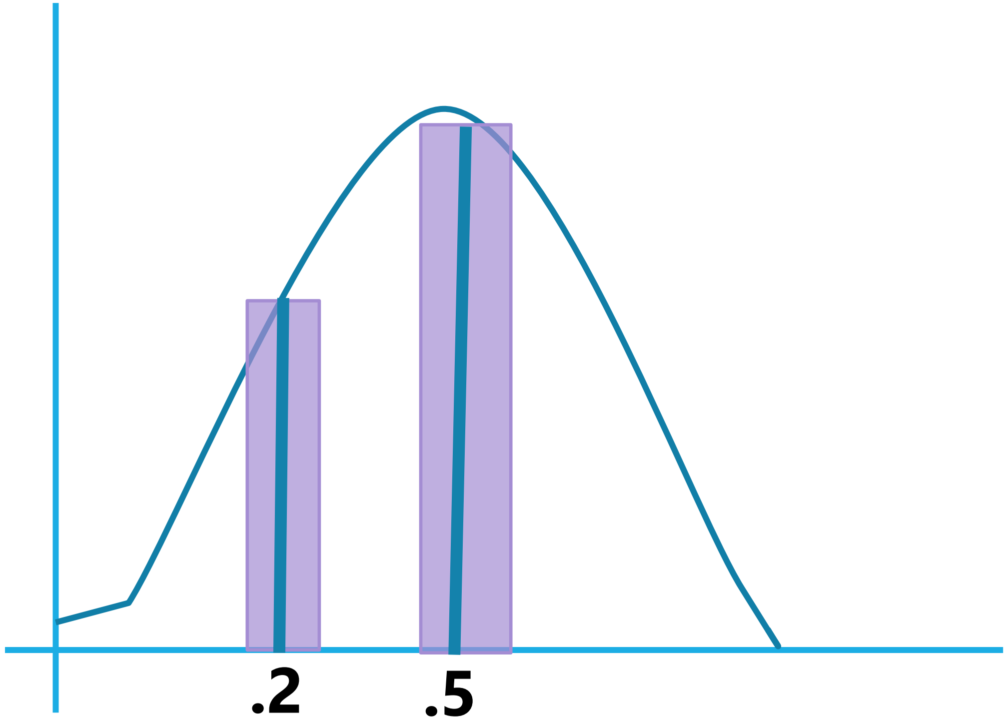

What happens if we look at the ratio

$$\frac{\mathbb{P}(X \approx .2)}{\mathbb{P}(X \approx .5)} = \frac{\mathbb{P}\left(.2 - \frac{\epsilon}{2} \leq X \leq .2 + \frac{\epsilon}{2}\right)}{\mathbb{P}\left(.5 - \frac{\epsilon}{2} \leq X \leq .5 + \frac{\epsilon}{2}\right)} = \frac{\epsilon f_X(.2)}{\epsilon f_X(.5)} = \frac{f_X(.2)}{f_X(.5)}$$

What exactly do the values in the PDF mean?

The number that when integrated over gives the probability of an event.

Equivalently, it's number such that:

-it is always non-negative

-integrating over all real numbers gives 1.

-comparing $f_X(k)$ and $f_X(\ell)$ gives the relative chances of $\boldsymbol{X}$ being near $\boldsymbol{k}$ or ℓ.

Cumulative Distribution Function (CDF)

What's a CDF?

The Cumulative Distribution Function $F_X(k) = \mathbb{P}(\boldsymbol{X} \leq \boldsymbol{k})$ analogous to the CDF for discrete variables.

$$F_X(k) = \mathbb{P}(X \leq k) = \int_{-\infty}^{k} f_X(z)\,\mathrm{d}z$$

So how do I get from CDF to PDF? Taking the derivative!

$$\frac{\mathrm{d}}{\mathrm{d}k} F_X(k) = \frac{\mathrm{d}}{dk}\left(\int_{-\infty}^{k} f_X(z)\,\mathrm{d}z\right) = f_X(k)$$

Comparing Discrete and Continuous

	Discrete Random Variables	Continuous Random Variables
Probability 0	Equivalent to impossible	All impossible events have probability 0, but not conversely.
Relative Chances	PMF: $p_X(k) = \mathbb{P}(X = k)$	PDF $f_X(k)$ gives chances relative to $f_X(k')$
Events	Sum over PMF to get probability	Integrate PDF to get probability
Convert from CDF to PMF	Sum up PMF to get CDF. Look for "breakpoints" in CDF to get PMF.	Integrate PDF to get CDF. Differentiate CDF to get PDF.
$\mathbb{E}[X]$	$\sum_{\omega} X(\omega) \cdot f_X(\omega)$	Next up!
$\mathbb{E}[g(X)]$	$\sum_{\omega} g\big(X(\omega)\big) \cdot f_X(\omega)$	Next up!
$\mathrm{Var}(X)$	$\mathbb{E}[X^2] - (\mathbb{E}[X])^2$	Next up!

Expectation and Variance

What about expectation?

For a **discrete** random variable X, we have $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$

For a **continuous** random variable X, we define:

$\mathbb{E}[X] = \int_{-\infty}^{\infty} z \cdot f_X(z)\, \mathrm{d}z$

Just replace summing over the PMF with integrating the PDF.

It still represents the average value of X.

Expectation of a function

For a **discrete** random variable X, we have $\mathbb{E}[g(X)] = \sum_{k \in \Omega_X} g(k) \cdot p_X(k)$

For any function g and any **continuous** random variable, X:

$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(z) \cdot f_X(z)\,\mathrm{d}z$

Again, analogous to the discrete case; just replace summation with integration and pmf with the pdf.

Linearity of Expectation

Still true!

$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y\} + c$

For all X, Y; even if they're continuous.

Won't show you the proof – for just $\mathbb{E}[aX + b]$, it's

$\mathbb{E}[aX + b] = \int_{-\infty}^{\infty}[aX(k) + b]f_X(k)\,\mathrm{d}k$

$= \int_{-\infty}^{\infty} aX(k)f_X(k)dk + \int_{-\infty}^{\infty} bf_X(k)dk$

$= a\int_{-\infty}^{\infty} X(k)f_X(k)dk + b\int_{-\infty}^{\infty} f_X(k)dk$

$= a\mathbb{E}[X] + b$

Variance

No surprises here

$$\mathbf{Var}(\boldsymbol{X}) = \mathbb{E}\left[\boldsymbol{X}^{\mathbf{2}}\right] - (\mathbb{E}[\boldsymbol{X}])^{\mathbf{2}} = \int_{-\infty}^{\infty} \boldsymbol{f}_{\boldsymbol{X}}(\boldsymbol{k})(\boldsymbol{k} - \mathbb{E}[\boldsymbol{X}])^{\mathbf{2}}\, \mathbf{d}\boldsymbol{k}$$

Comparing Discrete and Continuous

	Discrete Random Variables	Continuous Random Variables
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$\mathbb{E}[X]$	$\sum_{\omega} X(\omega) \cdot f_X(\omega)$	$\int_{-\infty}^{\infty} z \cdot f_X(z)\,\mathrm{d}z$
$\mathbb{E}[g(X)]$	$\sum_{\omega} g\big(X(\omega)\big) \cdot f_X(\omega)$	$\int_{-\infty}^{\infty} g(z) \cdot f_X(z)\,\mathrm{d}z$
$\mathrm{Var}(X)$	$\mathbb{E}[X^2] - (\mathbb{E}[X])^2$	$\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \int_{-\infty}^{\infty} (z - \mathbb{E}[X])^2 f_X(z)\,\mathrm{d}z$

Zoo of **Continuous** Random variables

and get practice with working with continuous random variables!

Continuous Zoo

This zoo defines common patterns for <u>continuous</u> random variables and gives us the PDF, CDF, expectation, and variance, so we don't have to compute it every time!

$X \sim \mathrm{Unif}(a,b)$	$X \sim \mathrm{Exp}(\lambda)$	$X \sim \mathcal{N}(\mu, \sigma^2)$
$f_X(k) = \frac{1}{b-a}$ for $a \le k \le b$ $F_X(k) = \frac{x-a}{b-a}$ if $a \le k < b$ $\mathbb{E}[X] = \frac{a+b}{2}$ $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$	$f_X(k) = \lambda e^{-\lambda k}$ for $k \ge 0$ $F_X(k) = 1 - e^{-\lambda k}$ if $k \ge 0$ $\mathbb{E}[X] = \frac{1}{\lambda}$ $\mathrm{Var}(X) = \frac{1}{\lambda^2}$	$f_X(k) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ $F_X(k) = \Phi\left(\frac{k-\mu}{\sigma}\right)$ $\mathbb{E}[X] = \mu$ $\mathrm{Var}(X) = \sigma^2$

It's a smaller zoo, but it's just as much fun! :D

Continuous Uniform Distribution

Scenario: Pick a random *real* number between a and b. X is our choice.
$>$ X is a **uniform random real number** between a and b -> $X \sim \mathrm{Unif}(a, b)$

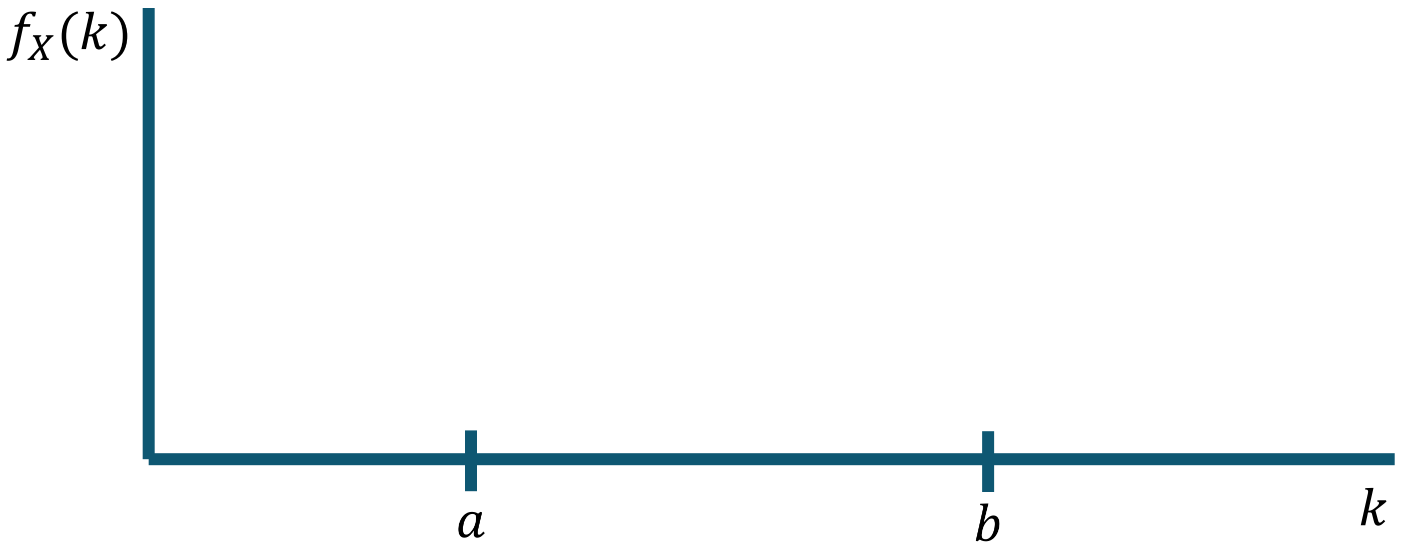
We saw the discrete uniform distribution before – it took on an integer between a *and* b
This is the **continuous** uniform distribution – it takes on a **real number** between a and b

Continuous Uniform Distribution - **PDF**

> X is a **uniform random real number** between a and b -> $X \sim \text{Unif}(a, b)$

What is the PDF, $f_X(k)$?

Draw a picture and think about the properties the PDF must have!



Fill out the poll everywhere:
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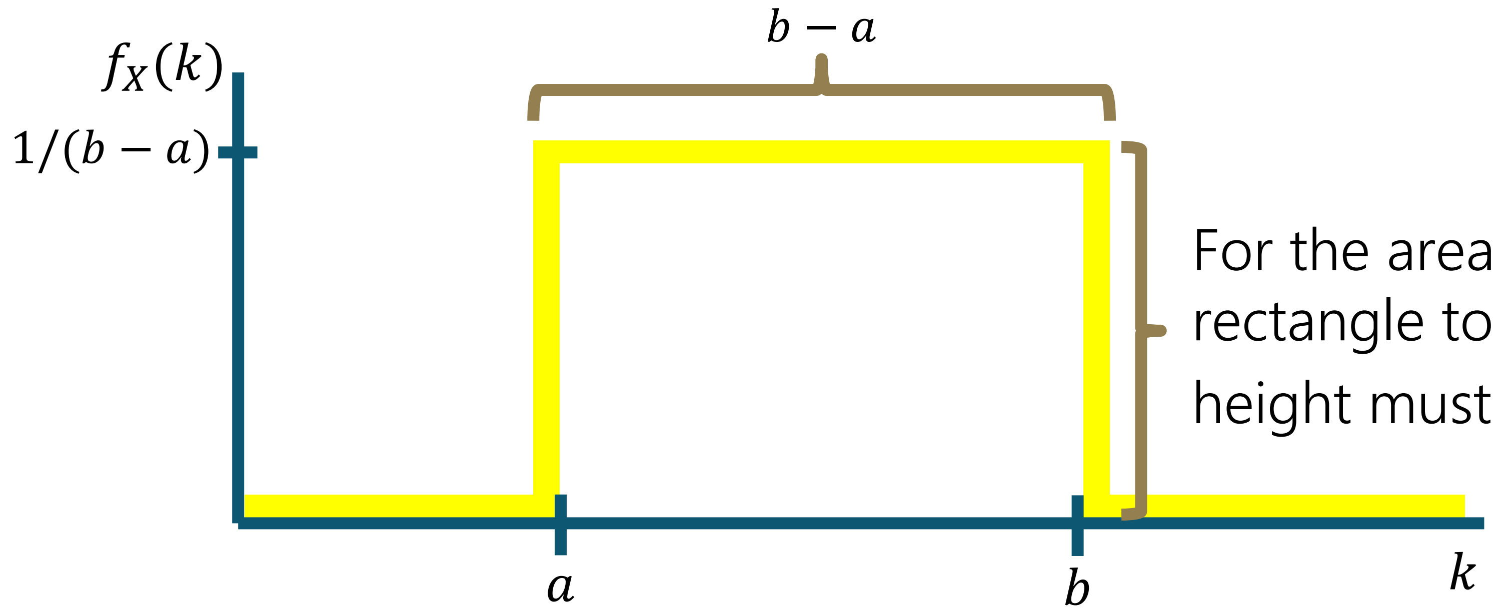


For the area of this rectangle to be 1, the height must be $\frac{1}{b-a}$

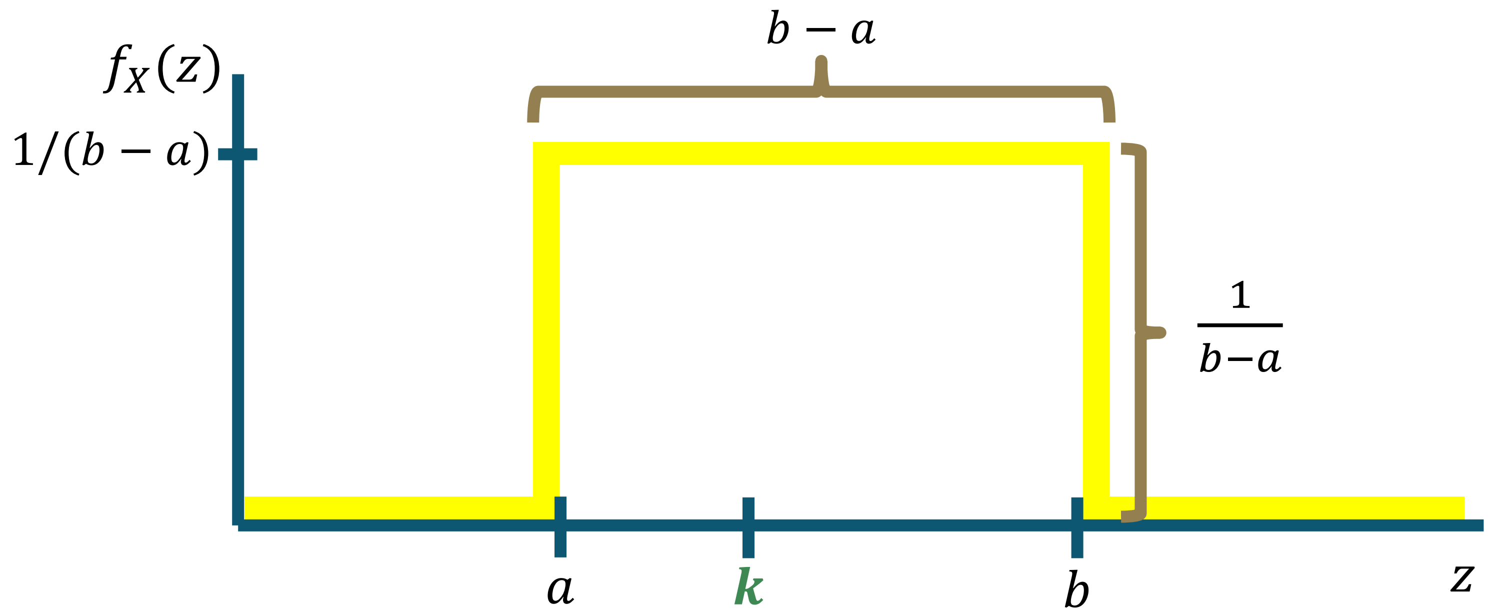
The area under a PDF must be 1. $\int_{-\infty}^{\infty} f_X(z)\, dz = 1$

$$f_X(k) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le k \le b \\ 0 & \text{otherwise} \end{cases}$$

Continuous Uniform Distribution – **CDF**(**Visual**)

> X is a **uniform random real number** between a and b -> $X \sim \text{Unif}(a, b)$

What is the CDF, $F_X(k)$? *Let's compute this "visually"….*



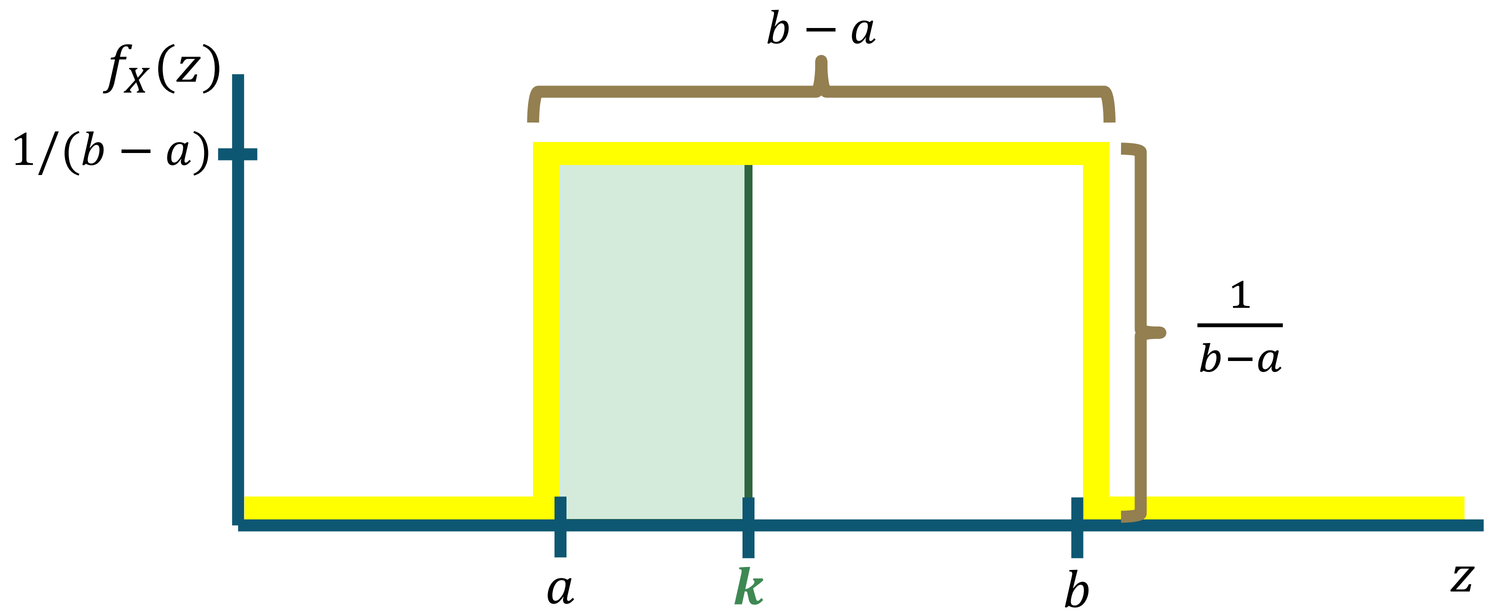
$$F_X(k) = \begin{cases} \\ \\ \end{cases}$$

Continuous Uniform Distribution – **CDF**(Visual)

> X is a **uniform random real number** between a and b -> $X \sim \text{Unif}(a, b)$

What is the CDF, $F_X(k)$? *Let's compute this "visually"….*

$F_X(k) = P(X \leq k)$ is the area of the green region below: $\frac{k-a}{b-a}$ if $a \leq k < b$



$$F_X(k) = \begin{cases} \dfrac{\boldsymbol{k-a}}{\boldsymbol{b-a}} & \textbf{if } \boldsymbol{a \leq k \leq b} \end{cases}$$

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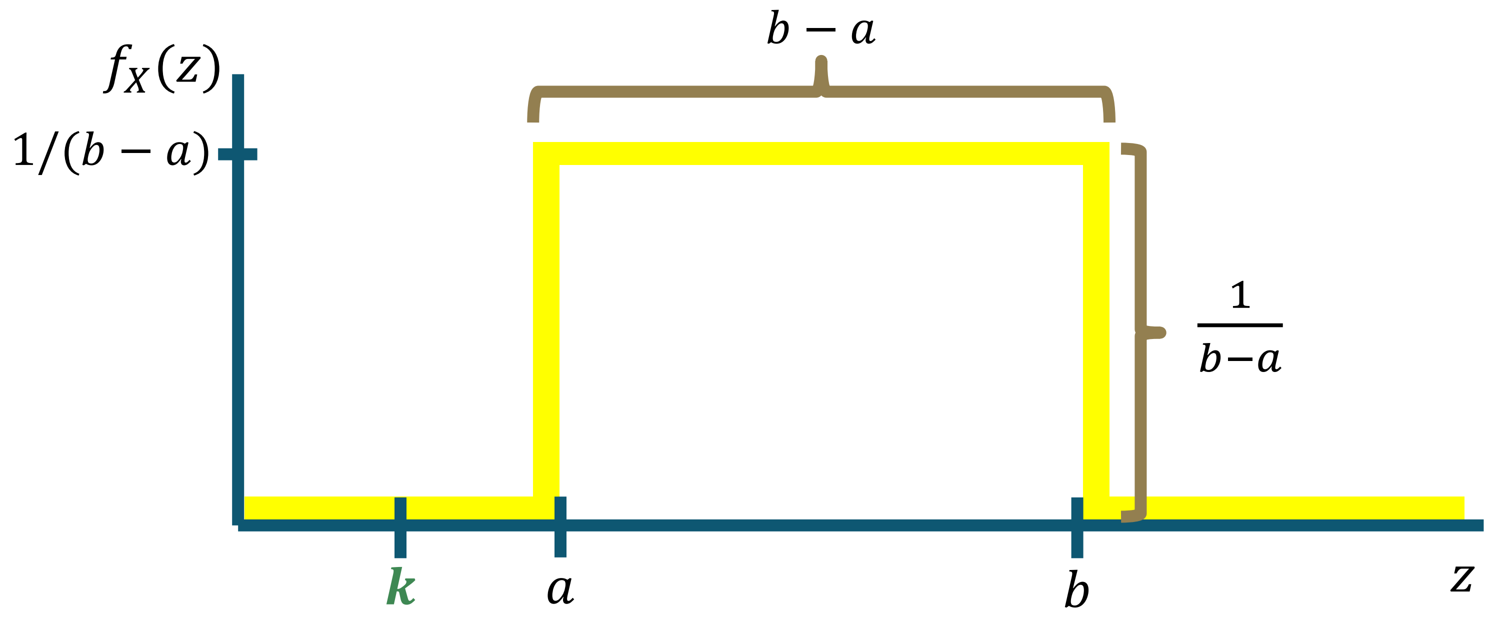

$$F_X(k) = \begin{cases} \mathbf{0} & \textbf{if } \boldsymbol{k < a} \\ \dfrac{k-a}{b-a} & \text{if } a \leq k \leq b \end{cases}$$

Continuous Uniform Distribution – **CDF**(Visual)

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What is the CDF, $F_X(k)$? *Let's compute this "visually"….*

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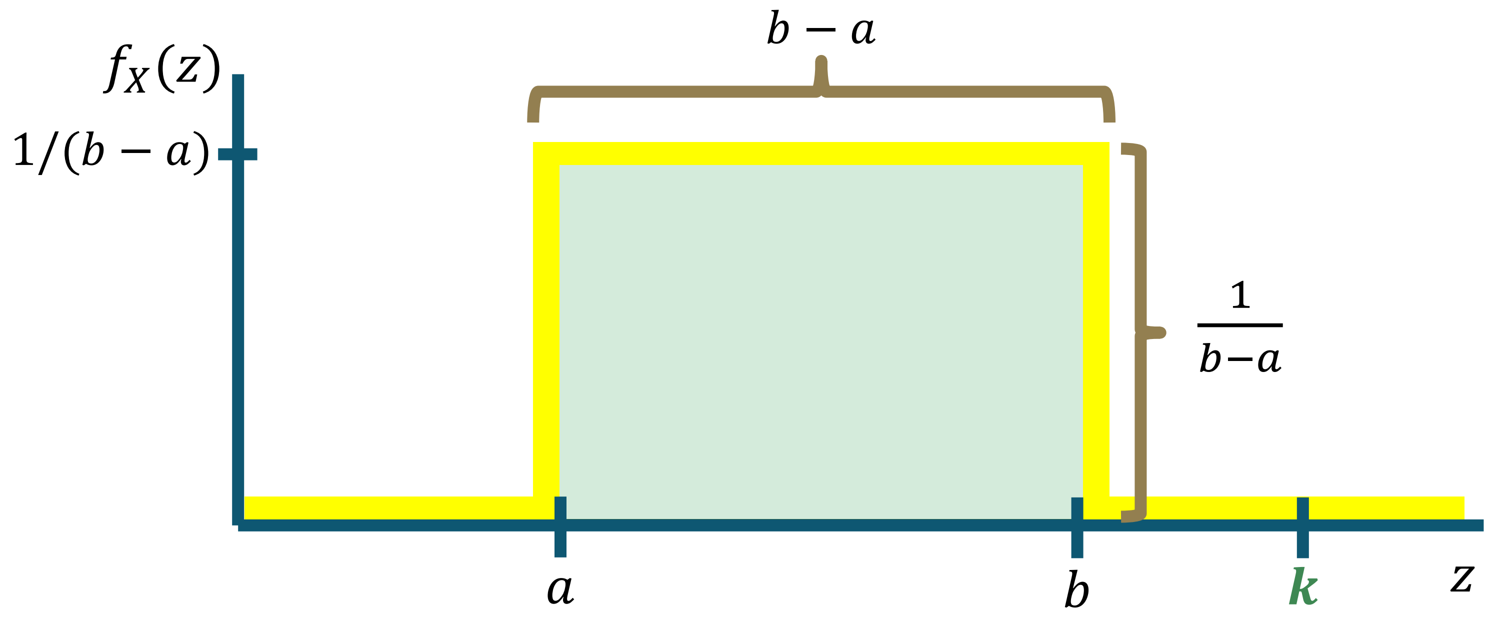

$$F_X(k) = \begin{cases} 0 & \text{if } k < a \\ \dfrac{k-a}{b-a} & \text{if } a \leq k \leq b \\ \mathbf{1} & \textbf{if } \boldsymbol{k \geq b} \end{cases}$$

Continuous <u>Uniform</u> Distribution – **CDF**(Integral)

> X is a **uniform random real number** between a and b -> $X \sim \text{Unif}(a, b)$

What is the CDF, $F_X(k)$? *Let's compute this "algebraically"....*

The CDF is: $F_X(k) = \int_{-\infty}^{k} f_X(z)dz$

We can compute this for the ranges based on what $\boldsymbol{k}$ is and what $\boldsymbol{f_X(z)}$ is for the values of $\boldsymbol{z}$ less than $\boldsymbol{k}$

Case when $\boldsymbol{k} \leq \boldsymbol{a}$: $F_X(k) = \int_{-\infty}^{k} f_X(z)dz = \int_{-\infty}^{k} 0\, dz = 0$

Case when $\boldsymbol{a} \leq \boldsymbol{k} \leq \boldsymbol{b}$: $F_X(k) = \int_{-\infty}^{k} f_X(z)dz = \int_{-\infty}^{a} 0\, dz + \int_{a}^{k} \frac{1}{b-a} dz = \frac{k-a}{b-a}$

Case when $\mathbf{k} > \boldsymbol{b}$: $F_X(k) = \int_{-\infty}^{k} f_X(z)dz = \int_{a}^{b} \frac{1}{b-a} dz + \int_{b}^{k} 0 dz = 1$

$$\boxed{F_X(k) = \begin{cases} 0 & \text{if } k < a \\ \dfrac{k-a}{b-a} & \text{if } a \leq k \leq b \\ \mathbf{1} & \textbf{if } \boldsymbol{k} \geq \boldsymbol{b} \end{cases}}$$

Let’s calculate an expectation

$$f_X(k) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq z \leq b \\ 0 & \text{otherwise} \end{cases}$$

Let X be a uniform random number between a and b.

$\mathbb{E}[X] = \int_{-\infty}^{\infty} z \cdot f_X(z)\,\mathrm{d}z$

$= \int_{-\infty}^{a} z \cdot 0\,\mathrm{d}z + \int_{a}^{b} z \cdot \frac{1}{b-a}\,\mathrm{d}z + \int_{b}^{\infty} z \cdot 0\,\mathrm{d}z$

$= 0 + \int_{a}^{b} \frac{z}{b-a}\,\mathrm{d}z + 0$

$= \frac{z^2}{2(b-a)}\Big|_{z=a}^{b} = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b^2-a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)} = \frac{a+b}{2}$

What about $\mathbb{E}[g(X)]$

Let $X \sim \text{Unif}(a, b)$, what about $\mathbb{E}[X^2]$?

$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} z^2 f_X(z) \mathrm{dz}$

$= \int_{-\infty}^{a} z^2 \cdot 0 \,\mathrm{dz} + \int_{a}^{b} z^2 \cdot \frac{1}{b-a} \mathrm{dz} + \int_{b}^{\infty} z^2 \cdot 0 \,\mathrm{dz}$

$= 0 + \int_{a}^{b} z^2 \cdot \frac{1}{b-a} \mathrm{dz} + 0$

$= \frac{1}{b-a} \cdot \frac{z^3}{3} \Big|_{z=a}^{b} = \frac{1}{b-a} \left(\frac{b^3}{3} - \frac{a^3}{3} \right) = \frac{1}{3(b-a)} \cdot (b-a)(a^2 + ab + b^2)$

$= \frac{a^2+ab+b^2}{3}$

Let’s assemble the variance

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{4(a^2+ab+b^2)}{12} - \frac{3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

Continuous Uniform Distribution

$X \sim \mathrm{Unif}(a, b)$ (uniform real number between a and b)

PDF: $f_X(k) = \begin{cases} \frac{1}{b-a} & \text{if } a \le k \le b \\ 0 & \text{otherwise} \end{cases}$

CDF: $F_X(k) = \begin{cases} 0 & \text{if } k < a \\ \frac{k-a}{b-a} & \text{if } a \le k \le b \\ 1 & \text{if } k \ge b \end{cases}$

$\mathbb{E}[X] = \frac{a+b}{2}$

$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$