Midterm Reminders

Instructions

- You have sixty minutes to complete this exam.
- You may not use a calculator or any other electronic devices during the exam.
- This is a closed note exam, with the exception of your own double-sided sheet of notes and a reference sheet with some of the most-used formulas has been provided.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- In general, show us the work you used to get to an answer; explanations will help us award partial credit, but we do not expect explanations at the level we usually require on homeworks.
- If your final answer to a question is correct, you will get full credit regardless of whether or not you provide any explanation.

Simplification Expectations

- Since you don't have a calculator for this exam, you do not have to do simplifications that could be done easily with a calculator. For example
$$\frac{\binom{5}{3} \cdot 17^2}{1-p} + 5^3$$
can be given as a final answer.
- However, answers which are much more complicated than the expected answer may receive deductions. For example: $\sum_{i=0}^{n} \binom{n}{i}$ or $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ are **not** simplified sufficiently.

Reference Sheet

Make sure to go over the reference sheet

Midterm - Reference Sheet

Complementary Counting: Let $\mathcal{U}$ be a (finite) universal set, and S a subset of interest. Then, $|S| = |\mathcal{U}| - |\mathcal{U} \setminus S|$.

Stars and Bars Method: The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Binomial Theorem: Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Principle of Inclusion-Exclusion (PIE):
2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
k events: singles - doubles + triples - quads + ...

Pigeonhole Principle: If there are n pigeons we want to put into k holes (where $n > k$), then at least one pigeonhole must contain at least 2 (or to be precise, $\lceil n/k \rceil$) pigeons.

Key Probability Definitions: The *sample space* is the set Ω of all possible outcomes of an experiment. An *event* is any subset $E \subseteq \Omega$. Events E and F are *mutually exclusive* if $E \cap F = \emptyset$.

Probability space: A *probablity space* is a pair $(\Omega, \mathbb{P})$, where Ω is the sample space and $\mathbb{P} : \Omega \to [0,1]$ is a *probability measure* such that $\sum_{x \in \Omega} \mathbb{P}(x) = 1$. The probability of an event $E \subseteq \Omega$ is $\mathbb{P}(E) = \sum_{x \in E} \mathbb{P}(x)$.

Conditional Probability: $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Bayes Theorem: $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$

Partition: Non-empty events $E_1, \ldots, E_n$ *partition* the sample space Ω if they are both:

- **(Exhaustive)** $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$ (they cover the entire sample space).
- **(Pairwise Mutually Exclusive)** For all $i \neq j$, $E_i \cap E_j = \emptyset$ (none of them overlap)

Law of Total Probability (LTP): If events $E_1, \ldots, E_n$ partition Ω, then for any event F:

$$\mathbb{P}(F) = \sum_{i=1}^{n} \mathbb{P}(F \cap E_i) = \sum_{i=1}^{n} \mathbb{P}(F \mid E_i)\,\mathbb{P}(E_i)$$

Chain Rule: Let $A_1, \ldots, A_n$ be events with nonzero probabilities. Then:

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_n \mid A_1 \cap \cdots \cap A_{n-1})$$

Independence: A and B are *independent* if any of the following equivalent statements hold:

1. $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$
2. $\mathbb{P}(A \mid B) = \mathbb{P}(A)$
3. $\mathbb{P}(B \mid A) = \mathbb{P}(B)$

Mutual Independence: We say n events $A_1, A_2, \ldots, A_n$ are *(mutually) independent* if, for *any* subset $I \subseteq [n] = \{1, 2, \ldots, n\}$, we have

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i)$$

This equation is actually representing 2^n equations since there are 2^n subsets of $[n]$.

Conditional Independence: A and B are *conditionally independent given an event C* if any of the following equivalent statements hold:

1. $\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)$
2. $\mathbb{P}(A \mid B \cap C) = \mathbb{P}(A \mid C)$
3. $\mathbb{P}(B \mid A \cap C) = \mathbb{P}(B \mid C)$

Random Variable (RV): A random variable (RV) X is a numeric function of the outcome $X : \Omega \to \mathbb{R}$. The set of possible values X can take on is its *range/support*, denoted Ω_X.
If Ω_X is finite or countable infinite (typically integers or a subset), X is a *discrete RV*.

Probability Mass Function (PMF): For a discrete RV X, assigns probabilities to values in its range. That is $p_X : \Omega_X \to [0,1]$ such that: (1) $p_X(k) \geq 0$ for all $k \in \Omega_X$; (2) $\sum_{k \in \Omega_X} p_X(k) = 1$.
Furthermore, $p_X(k) = \mathbb{P}(X = k)$.

Cumulative Distribution Function (CDF): The *cumulative distribution function (CDF)* of ANY random variable (discrete or continuous) is defined to be the function $F_X : \mathbb{R} \to \mathbb{R}$ with $F_X(t) = \mathbb{P}(X \leq t)$.

Expectation (Discrete): $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$.

Linearity of Expectation (LoE):
For any random variables X, Y (possibly dependent): $\mathbb{E}[aX + bY + c] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y] + c$

Law of the Unconscious Statistician (LOTUS):
For a RV X and function g: If X is discrete, $\mathbb{E}[g(X)] = \sum_{b \in \Omega_X} g(b) \cdot p_X(b)$.

Variance: $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

Property of Variance: $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.

Variance Adds for Independent RVs: If X, Y are independent, then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

Similar format with quizzes

MCQ:

For the questions below,

- Questions with ◯ Circles have exactly one best answer. **Fully fill in** the circle for the one best answer.
- Questions with ☐ squares are "mark all that apply" questions. **Fully fill in** the square for all correct options.

Final answers in the box

True / false

For any events E and F such that $\Pr(E|E \cap F) > 0$, it holds that $\Pr(E|E \cap F) \leq \Pr(E|F)$

Counting

Marlene wants to buy 20 bubble teas for a party. There are 4 flavors available- brown sugar, matcha, lychee, and strawberry.

If she wants to buy at least 2 of each flavor, and if bubble teas of the same flavor are indistinguishable, how many ways are there to select the 20 drinks?

Counting

Marlene wants to buy 20 bubble teas for a party. There are 4 flavors available- brown sugar, matcha, lychee, and strawberry.

How many ways can Marlene order 1 drink, such that the drink she ordered contains brown sugar flavor, and/or contains matcha flavor?

Consider two orders the same they have the same combination of flavors.

Probability

A professor has a test bank of 20 questions that she will draw on for a particular exam. A particular student knows how to solve 12 of them. The exam she gives is a random subset of 8 of the questions.

What is the probability that the student knows how to solve all 8 problems?

Probability

A professor has a test bank of 20 questions that she will draw on for a particular exam. A particular student knows how to solve 12 of them. The exam she gives is a random subset of 8 of the questions.

What is the probability that the student knows how to solve exactly 6 of the problems?

Expectation

A professor has a test bank of 20 questions that she will draw on for a particular exam. A particular student knows how to solve 12 of them. The exam she gives is a random subset of 8 of the questions.

What is the expected number of questions the student will get right?

Conditional Probability

Allie and her daughter Sophia love watching Kraken ice hockey games. For some games, one of them will wear the lucky tentacle, which affects the Kraken's chances of winning.

- If neither wears the lucky tentacle, the Kraken win with probability 1/2.
- If Allie wears the lucky tentacle, the Kraken win with probability p.
- If Sophia wears the lucky tentacle, the Kraken win with probability 2p.

Sophia has an early bed-time, so she only wears the tentacle 1/5 of the time. Allie wears it 2/5 of the time and the remaining 2/5 neither do.

Note that these are the only possibilities (there is no way for both to wear the one lucky tentacle). Let A, S, N be the events that Allie, Sophia, or neither wear the lucky tentacle (respectively).

Let W be the event that the Kraken win.

Conditional Probability

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Which conditional probability or probability is referred to in the first bullet?

Conditional Probability

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Let W be the event that the Kraken win.

If the Kraken won, what is the probability that Sophia was wearing the lucky tentacle?

Write a formula to represent this probability using only notation (no numbers nor expressions using p; just probabilities, conditional probabilities, and events) that describe the situation.

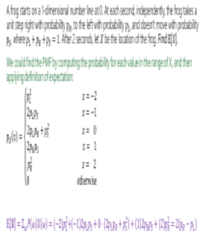
Frogger

A frog starts on a 1-dimensional number line at 0.

Each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$.

After 2 seconds, let X be the location of the frog. **Find** $\mathbb{E}[X]$.

Frogger – Brute Force



A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. **Find $\mathbb{E}[X]$.**

We could find the PMF by computing the probability for each value in the range of X, and then applying definition of expectation:

$$p_X(x) = \begin{cases} p_L^2 & x = -2 \\ 2p_L p_S & x = -1 \\ 2p_L p_R + p_S^2 & x = \ \ 0 \\ 2p_R p_S & x = \ \ 1 \\ p_R^2 & x = \ \ 2 \\ 0 & \text{otherwise} \end{cases}$$

We think about the outcomes that correspond to each value of X and compute the probability of that. For example, X=0 happens when the frog is at the same position after 2 sec – this means it either moved left and then right, or right and then left, or did not move both seconds.

$$\mathbb{E}[\boldsymbol{X}] = \Sigma_\omega P(\omega)X(\omega) = (-2)p_L^2 + (-1)2p_L p_S + 0 \cdot (2p_L p_R + p_S^2) + (1)2p_R p_S + (2)p_R^2 = 2(p_R - p_L)$$

Frogger – LOE

Or we can apply LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. **Find $\mathbb{E}[X]$.**

Define X_i as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{2} X_i\right] = \sum_{i=1}^{2} \mathbb{E}[X_i] = 2(p_R - p_L)$$

Frogger – LOE

If we interested in a whole minute (60 sec), the first approach would be awful because we would need to compute many probabilities or deal with a gnarly summation! Instead, we can use LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 60 seconds, let X be the location of the frog. **Find** $\mathbb{E}[X]$.

Define X_i as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{60} X_i\right] = \sum_{i=1}^{60} \mathbb{E}[X_i] = \mathbf{60}(p_R - p_L)$$