Linearity of Expectation

CSE 312 26Su

Lecture 9

Announcements

- Midterm information has been posted on the course website
- Quiz 2 grades are released
- HW 3 is due this week on Wednesday

Where Are We?

Last time:

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Today:

- Review Expectation
- Linearity of Expectation

Expectation

Expectation

The "expectation" (or "expected value") of a random variable X is:

$$\mathbb{E}[\boldsymbol{X}] = \sum_{\boldsymbol{k} \in \Omega_X} \boldsymbol{k} \cdot \mathbb{P}(\boldsymbol{X} = \boldsymbol{k})$$

$$\mathbb{E}[\boldsymbol{X}] = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\omega)$$

Intuition: The weighted average of values X could take on.

Weighted by the probability you actually see them.

Example 2

You roll a biased die.

It shows a 6 with probability $\frac{1}{3}$, and 1,…,5 with probability $2/15$ each.

Let X be the value of the die. What is $\mathbb{E}[X]$?

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Let X be the value of the die. What is $\mathbb{E}[X]$?

1. Write the PMF for X: $\mathrm{p}_X(k) = \begin{cases} 2/15 & k \in \{1,2,3,4,5\} \\ 1/3 & k = 6 \\ 0 & \text{otherwise} \end{cases}$

2. Plug in formula: $\mathbb{E}[X] = \frac{2}{15} \cdot 1 + \frac{2}{15} \cdot 2 + \frac{2}{15} \cdot 3 + \frac{2}{15} \cdot 4 + \frac{2}{15} \cdot 5 + \frac{1}{3} \cdot 6 = 4$

$\mathbb{E}[X]$ is not just the most likely outcome!

Linearity of Expectation

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}]=\mathbb{E}[\boldsymbol{X}]+\mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}] = \mathbb{E}[\boldsymbol{X}] + \mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

Proof:

$$\begin{aligned}\mathbb{E}[X+Y] &= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)\big(X(\omega)+Y(\omega)\big)\\ &= \Sigma_{\omega\in\Omega}(\mathbb{P}(\omega)X(\omega)+\mathbb{P}(\omega)Y(\omega))\\ &= \Sigma_{\omega\in\Omega}\mathbb{P}(\omega)X(\omega)+\Sigma_{\omega\in\Omega}\mathbb{P}(\omega)Y(\omega)\\ &= \mathbb{E}[X]+\mathbb{E}[Y]\end{aligned}$$

Definition of expectation:

$$\mathbb{E}[\boldsymbol{X}] = \sum_{\omega\in\Omega} X(\omega)\cdot\mathbb{P}(\omega)$$

Linearity of Expectation

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Note: X and Y do not have to be independent

Extending this to n random variables, $X_1, X_2, \dots, X_n$

$$\mathbb{E}[X_1+X_2+\cdots+X_n]=\mathbb{E}[X_1]+\mathbb{E}[X_2]+\cdots+\mathbb{E}[X_n]$$

This can be proven by induction.

Linearity of Expectation

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For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}]=\mathbb{E}[\boldsymbol{X}]+\mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

Constants are also fine:

For real numbers a, b, c

$$\begin{aligned}\mathbb{E}[aX+bY+c] &= \mathbb{E}[aX]+\mathbb{E}[bY+c] \\ &= a\mathbb{E}[X]+b\mathbb{E}[Y]+c\end{aligned}$$

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$

- How many fish do both of you bring on an average day?

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$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

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- You can sell each for \$10 per fish, but you need \$15 (total) for expenses. What is your average profit?

$$\mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15 = 100 - 15 = 85$$

Coin Tosses

If we flip a coin twice, what is the expected number of heads that come up?

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If we flip a coin twice, what is the expected number of heads that come up?

Let Y be the r.v. representing the total number of heads

$$p_Y(y) = \begin{cases} \frac{1}{4} & \text{if } y = 0 \\ \frac{1}{2} & \text{if } y = 1 \\ \frac{1}{4} & \text{if } y = 2 \\ 0 & otherwise \end{cases}$$

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$$\mathbb{E}[Y] = \Sigma_{k \in \Omega_Y} p_Y(k) \cdot k = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let X be the r.v. representing the total number of heads.

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Make a prediction --- what should $\mathbb{E}[X]$ be?

a) $n + p$
b) p^n
c) np
d) n/p

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let X be the r.v. representing the total number of heads.

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(X = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1 - p)^{n-k}$$

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$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}$$

$$= np(p + (1-p))^{n-1} = np$$

Binomial Theorem!

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity.

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(Y = k) = \sum^{n} \ldots p)^{n-k}$$

$$= \sum^{n} \ldots$$

$$\ldots_{k=1} n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}$$

$$= np(p + (1-p))^{n-1} = np$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Binomial Theorem!

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

No one wants to do proofs like this every time!

Linearity of Expectation

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$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}] = \mathbb{E}[\boldsymbol{X}] + \mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

Extending this to n random variables, $X_1, X_2, \dots, X_n$

$$\mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

This can be proven by induction.

Indicator Random Variables

For any event A, we can define the indicator random variable $\mathbf{1}[A]$ for A

$$\mathbf{1}[A] = X = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X = 1) = \mathbb{P}(A)$$
$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(A)$$

$$p_X(k) = \begin{cases} \mathbb{P}(A) & \text{if } k = 1 \\ 1 - \mathbb{P}(A) & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} &\mathbb{E}[X] \\ &= 1 \cdot p_X(1) + 0 \cdot p_X(0) \\ &= p_X(1) = \mathbb{P}(A) \end{aligned}$$

You'll also see notation like:

$\mathbb{1}[A]$, 1_A, $\mathbb{1}[\text{some boolean}]$

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

What indicators can we define? What ‘Booleans’ have enough information to combine (add) and solve the problem?

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if the ith coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

$$\mathbb{P}(X_i = 1) = p$$
$$\mathbb{P}(X_i = 0) = 1 - p$$

Let X be the total number of heads

Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if the ith coin flip is heads} \\ 0 & \text{otherwise} \end{cases} \longrightarrow X = \sum_{i=1}^{n} X_i$$

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$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

Repeated Coin Tosses (Again)

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$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

By **<u>Linearity of Expectation</u>**,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} p = np$$

Computing complicated expectations

We often use these three steps to solve complicated expectations

1. **Decompose:** Finding the right way to decompose the random variable into sum of simple random variables
$$X = X_1 + X_2 + \cdots + X_n$$
2. **LOE:** Apply Linearity of Expectation
$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$
3. **Conquer:** Compute the expectation of each X_i

Often X_i are indicator random variables

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose: Let X be the number of pairs with the same birthday

LOE:

Conquer:

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose: Let X be the number of pairs with the same birthday

Define X_{ij} as follows:

$$X_{ij} = \begin{cases} 1 & \text{if person i, j have the same bithday} \\ 0 & \text{otherwise} \end{cases} \qquad X = \Sigma_{i,j} X_{ij}$$

LOE:

Conquer:

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<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}\left[\Sigma_{i,j} X_{ij}\right] = \Sigma_{i,j} \mathbb{E}\left[X_{ij}\right]$$

<u>Conquer:</u>

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LOE:

$$\mathbb{E}[X] = \mathbb{E}\left[\Sigma_{i,j} X_{ij}\right] = \Sigma_{i,j}\mathbb{E}\left[X_{ij}\right]$$

Conquer:

$$\mathbb{E}\left[X_{ij}\right] = \mathbb{P}\left(X_{ij} = 1\right) = \frac{365}{365 \cdot 365} = \frac{1}{365}$$

$$\mathbb{E}[X] = \binom{m}{2} \cdot \mathbb{E}\left[X_{ij}\right] = \binom{m}{2} \cdot \frac{1}{365}$$

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and n-1 (equally likely)

Let X be the number of people that end up in front of their own name tag. **Find $\mathbb{E}[X]$.**

Decompose:

What X_i can we define that have the needed information?

LOE:

Conquer:

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

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X is the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

<u>Decompose:</u> Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person i sits infront of their own name tag} \\ 0 & \text{otherwise} \end{cases}$$

Note: $X = \Sigma_{i=1}^{n} X_i$

<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} \mathbb{E}[X_i]$$

<u>Conquer:</u>

These X_i are not independent! That's ok!!

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<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} \mathbb{E}[X_i]$$

<u>Conquer:</u>

$$\mathbb{E}[X_i] = P(X_i = 1) = \frac{1}{n-1} \qquad \mathbb{E}[X] = n \cdot \mathbb{E}[X_i] = \frac{n}{n-1}$$

How to know what indicators to define?

- "find the expected number of students (from n in total) who show up in class"
  -> maybe define indicator RVs $\mathbf{X_i}$ that is $\mathbf{1}$ if the i'th student shows up
  -> $\mathbf{X = \sum_{i=1}^{n} X_i = X}$
- "find the expected number of defective items in a batch of 100 items"
  -> maybe define indicator RVs $\boldsymbol{X_i}$ that is $\mathbf{1}$ if the i'th item is defective
  -> $\boldsymbol{X = \sum_{i=1}^{n} X_i}$
- "find the expected number of A that have B (B is some property)“
  -> maybe define indicator RVs $\boldsymbol{X_i}$ for the i'th possible A that is 1 if that A has B

Techniques For Finding Expectation

1. Definition of Expectation

When the support is small enough and we can find the PMF of X

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot \mathbb{P}(X = k)$$

2. Linearity of Expectation

When we can break X into a sum

Break into indicator RVs if it's some kind of count

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[\mathrm{Y}]$$

3. LOTUS (Law of the unconscious statistician)

When it's a function of X (e.g, X2)

$$\mathbb{E}[g(X)] = \sum_{k \in \Omega_X} g(k) \cdot \mathbb{P}(X = k)$$

Where are we?

A *random variable* is a way to summarize what outcome you saw.

Ω is set of possible outcomes

Outcome $\boldsymbol{\omega \in \Omega}$ $\rightarrow$ X $\rightarrow$ **Quantitative value**

Support (set of values RV can take),

Probability Mass Function & Cumulative Distribution Function **describe** probabilities

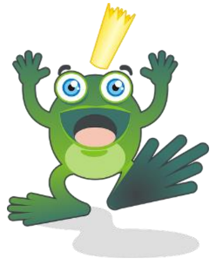
The expectation of a RV is its average value. (summarizes a RV)

> Expectation is **linear:** $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

- $X + Y$ is a random variable – it's a function that outputs a number given an outcome (or, here, a combination of outcomes).

Extra Practice

Frogger

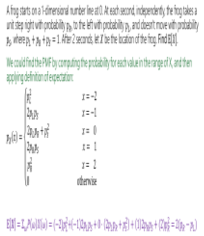


A frog starts on a 1-dimensional number line at 0.

Each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$.

After 2 seconds, let X be the location of the frog. **Find** $\mathbb{E}[X]$.

Frogger – Brute Force



A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. **Find $\mathbb{E}[X]$.**

We could find the PMF by computing the probability for each value in the range of X, and then applying definition of expectation:

$$p_X(x) = \begin{cases} p_L^2 & x = -2 \\ 2p_L p_S & x = -1 \\ 2p_L p_R + p_S^2 & x = 0 \\ 2p_R p_S & x = 1 \\ p_R^2 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

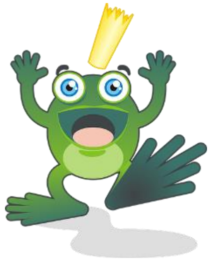
We think about the outcomes that correspond to each value of X and compute the probability of that. For example, X=0 happens when the frog is at the same position after 2 sec – this means it either moved left and then right, or right and then left, or did not move both seconds.

$$\mathbb{E}[\boldsymbol{X}] = \Sigma_\omega P(\omega)X(\omega) = (-2)p_L^2 + (-1)2p_L p_S + 0 \cdot (2p_L p_R + p_S^2) + (1)2p_R p_S + (2)p_R^2 = 2(p_R - p_L)$$

Frogger – LOE

Or we can apply LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. Find $\mathbb{E}[X]$.



Define X_i as follows:

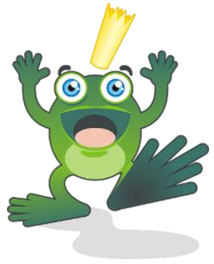
$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{2} X_i\right] = \sum_{i=1}^{2} \mathbb{E}[X_i] = 2(p_R - p_L)$$

Frogger – LOE



If we interested in a whole minute (60 sec), the first approach would be awful because we would need to compute many probabilities or deal with a gnarly summation! Instead, we can use LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R, to the left with probability p_L, and doesn't move with probability p_S, where $p_L + p_R + p_S = 1$. After 60 seconds, let X be the location of the frog. **Find** $\mathbb{E}[X]$.

Define X_i as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{60} X_i\right] = \sum_{i=1}^{60} \mathbb{E}[X_i] = 60(p_R - p_L)$$