# Random Variables

CSE 312 26Su

Lecture 8

# Quiz 2

# Where Are We?

Last time:

- Review
  - Counting
  - Uniform Probability Space
  - Conditional Probability
- Intro to Random Variables

Today:

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

# Defining Events Can Be *Tedious*

For example, if we're interested in analyzing the sum of 2 random dice…

We might define events for all the possible outcomes:

$E_1$ ~ the sum is 1

$E_2$ ~ the sum is 2

$E_3$ ~ the sum is 3

…

$E_{12}$ ~ the sum is 12

# Defining and Using Events Can Be *Tedious*

For example, if we're interested in analyzing the sum of 2 random dice…

Now, how might we express the event that the sum is more than 6?

We could define an event $A \sim \textbf{the sum is more than 6}$ or since that's a bit undescriptive we might look for $P(E_7 \cup E_8 \cup E_9 \cup E_{10} \cup E_{11} \cup E_{12})$

We want a way to easily say something "[the sum of the dice] > 6" or be able to summarize things like "what is [the sum of the dice] on average?"

# Random Variables

# Random Variable

Informally: A random variable is a way to **summarize** the important (numerical) information from your outcome.

**Random Variable**

**$X: \Omega \to \mathbb{R}$ is a random variable**
**$X(\omega)$ is the summary of the outcome $\omega$**

Formally: *Function* assigning a value to outcomes of a random experiment

$\Omega$ is set of possible outcomes

**Outcome $\boldsymbol{\omega \in \Omega}$** $\rightarrow$ $X$ $\rightarrow$ **Quantitative value**

# The sum of two dice

## EVENTS

We could define

$E_2 =$"sum is 2"

$E_3 =$"sum is 3"

…

$E_{12} =$"sum is 12"

And ask "which event occurs"?

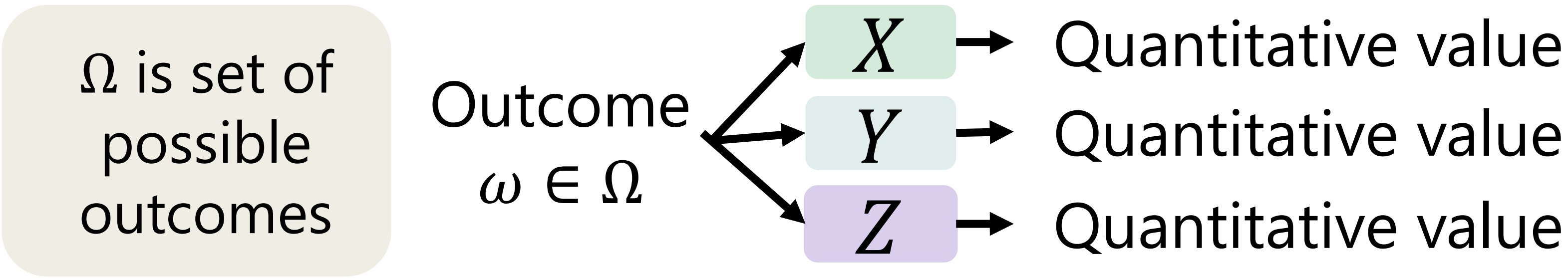
## RANDOM VARIABLE

$X: \Omega \to \mathbb{R}$

$X$ is the sum of the two dice.

# More random variables

From one sample space, you can define many random variables.




E.g., Roll a fair red die and a fair blue die, $\Omega$ is the set of possible outcomes

Let $D$ be the value of the red die minus the blue die $D(4,2) = 2$

Let $S$ be the sum of the values of the dice $S(4,2) = 6$

Let $M$ be the maximum of the values $M(4,2) = 4$

…

# Notational Notes

> We will always use capital letters for random variables.

> It's common to use lower-case letters for the values they could take on.

> When we say $X = 2$, we are referring to the set of outcomes that the random variable $X$ assigns the value 2

For example, if $X$ is the number of heads in three coin flips, $X = 2$ corresponds to the set of outcomes $\{HHT, HTH, THH\}$

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*Note that $X = 2$ is a set of outcomes, so it is an event!*

# How do we describe random variables?

Random variable gives a *quantitative property* of an outcome in a random experiment.

e.g., the number of coin flips till the first head, or the sum of two dice rolls

> **Support**

e.g., what are the possible “number of coin flips” it could possibly take?

> **Probability Mass Function**

e.g., what’s the probability it takes 3 coin flips till the 1st head? what about 5 flips?

> **Cumulative Distribution Function**

e.g., what’s the probability it takes *less than* 3 coin flips till the 1st head?

> **Expectation**

e.g., how many coin flips can we *expect* it to take till the first head <u>on average</u>?

> **Variance**

e.g., on average, how much does the “number of coin flips” deviate from the expectation?

# Support

The "support" (aka "the range") is the set of values $X$ can actually take.
We called this the "image" in 311.

E.g., We roll a red and a blue dice and define these 3 random variables:

$D$ (difference of red and blue dice) has support $\{-5, -4, -3, \ldots, 4,5\}$

$S$ (sum) has support $\{2,3, \ldots, 12\}$

$M$ (max of the two dice) has support ________________

***Each value in the support corresponds to some outcome(s) from Ω***

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$S$ (sum) has support $\{2,3, \dots, 12\}$

$M$ (max of the two dice) has support $\{1,2,3,4,5,6\}$

# Probability Mass Function

⚠ A random variable $X$ is <u>**not**</u> an event.

But, $\boldsymbol{X = 2}$ **is an event** – it's the event/set of outcomes where the random variable takes on the value 2

Often we're interested in the event $\{\omega : X(\omega) = k\}$

Which is the event…that $X = k$.

e.g., $S = 7$ is the event that two dice sum to 7

We'll write $\mathbb{P}(X = k)$ to describe the probability of that event

so $\mathbb{P}(S = 2) = \frac{1}{36}$, $\mathbb{P}(S = 7) = \frac{6}{36}$

The function that tells you $\mathbb{P}(X = k)$ is the "probability mass function"

We'll often write $\mathrm{p}_X(k) = \mathbb{P}(X = k)$ for the PMF.

# Probability Mass Function

Let $T$ be the number of 2's rolling a (fair) red and blue die.

*What is the range of $T$?* $\Omega_T = \{0,1,2\}$

*What is the PMF of $T$?*

$p_T(0) = 25/36$, $p_T(1) = 10/36$, $p_T(2) = 1/36$

$$p_T(k) = \begin{cases} 25/36 & k = 0 \\ 10/36 & k = 1 \\ 1/36 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

> Often use **piecewise function** to give probabilities for different values of k
> This is a function, so include **otherwise case**, so it is defined for all values

# Partition

A random variable partitions $\Omega$.

$$\sum_{k \in \Omega_x} p_X(k) = 1$$

Let $T$ be the number of 2's rolling a (fair) red and blue die.

$p_T(0) = 25/36$

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| | D2=1 | D2=2 | D2=3 | D2=4 | D2=5 | D2=6 |
|---|---|---|---|---|---|---|
| D1=1 | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1.6) |
| D1=2 | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| D1=3 | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| D1=4 | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| D1=5 | (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
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# Try It Yourself

There are 20 balls, numbered 1,2,...,20 in an urn.

You'll draw out a size-three subset. (i.e. without replacement)

$\Omega = \{$size three subsets of $\{1, \dots, 20\}$ $\}$, $\mathbb{P}(\cdot)$ is uniform measure.

Let $X$ be the largest value among the three balls.
e.g., if the outcome is $\{4,2,10\}$ then $X = 10$.

> What is the **support of $\boldsymbol{X}$**?

> Write down the **PMF of $\boldsymbol{X}$**
i.e., what is $p_X(k) = \mathbb{P}(X = k)$?

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Let $X$ be the largest value among the three balls.

$\Omega_X =$

$$p_X(k) = \begin{cases} & \\ & \end{cases}$$

# Try It Yourself

There are 20 balls, numbered 1,2,…,20 in an urn.

You'll draw out a size-three subset. (i.e. without replacement)

Let $X$ be the largest value among the three balls.

$$\Omega_X = \{3,4,5, \dots, 19, 20\}$$

$$p_X(k) = \begin{cases} \binom{k-1}{2} / \binom{20}{3} & \text{if } k \in \mathbb{N},\ 3 \leq k \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

*Good checks:*

if you sum up $p_X(k)$ do you get 1?

is $p_X(k) \geq 0$ for all $k$? Is it defined for all $k$?

# Cumulative Distribution Function (CDF)

The **PMF gives the probability $\boldsymbol{X = k}$**
*(and is the most common way to describe a random variable)*

There's a second representation:

The **cumulative distribution function (CDF) gives the probability $\boldsymbol{X \leq k}$**

More formally, $\mathbb{P}(\{\omega: X(\omega) \leq k\})$

Often written $\boldsymbol{F_X(k)} = \mathbb{P}(\boldsymbol{X \leq k})$

$F_X(k) = \sum_{i: i \leq k} p_X(i)$
"sum up the probabilities of $X$ taking all possible numbers less than $k$"

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Let $X$ be the largest value among the three balls.

$\Omega_X = \{3,4,5, \dots, 19, 20\}$

$$F_X(k) = \mathbb{P}(X \le k) = \begin{cases} \\ \\ \end{cases}$$

Think "what is the probability the largest value is less than or equal to $10$?"

# Try it yourself

There are 20 balls, numbered 1,2,…,20 in an urn.

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Let $X$ be the largest value among the three balls.

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$$F_X(k) = \mathbb{P}(X \leq k) = \begin{cases} 0 & \text{if } k < 3 \\ \binom{\lfloor k \rfloor}{3} / \binom{20}{3} & \text{if } 3 \leq k \leq 20 \\ 1 & \text{otherwise} \end{cases}$$

Think "what is the probability the largest value is less than or equal to $k$?"

# Try it yourself

There are 20 balls, numbered 1,2,…,20 in an urn.

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Let $X$ be the largest value among the three balls.

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*Good checks:*

Is $F_X(\infty) = \mathbb{P}(X \leq \infty) = 1$? If not, something is wrong.

Is $F_X(k)$ increasing? If not something is wrong.

Is $F_X(k)$ defined for all real number inputs? If not something is wrong.

# Summary (PMF and CDF)

## PROBABILITY MASS FUNCTION

Defined for all $\mathbb{R}$ inputs.

Usually has "$0$ otherwise" as an extra case.

$\sum_x p_X(x) = 1$

$0 \leq p_X(x) \leq 1$

$\sum_{i:i \leq k} p_X(k) = F_X(k)$

## CUMULATIVE DISTRIBUTION FUNCTION

Defined for all $\mathbb{R}$ inputs.

Often has "$0$ otherwise" and $1$ otherwise" extra cases

Non-decreasing function

$0 \leq F_X(k) \leq 1$

$\lim_{x \to -\infty} F_X(k) = 0$

$\lim_{x \to \infty} F_X(k) = 1$

# Expectation

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## Expectation

The "expectation" (or "expected value") of a random variable $X$ is:

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot \mathbb{P}(X = k)$$

Intuition: The **weighted average of values $X$ could take on**.

Weighted by the probability you actually see them.

e.g., if $Y \sim$num. flips to get a head,

$E[Y]$ is num. of flips we expect it to take *on average*

# Example 1

Flip a fair coin twice (independently).

What is the *expected number of heads* we see?

Let $X$ be the number of heads.

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Flip a fair coin twice (independently).
What is the *expected number of heads* we see?

Let $X$ be the number of heads.

$\Omega = \{TT, TH, HT, HH\}$, $\mathbb{P}()$ is uniform measure.

$\Omega_X = \{0,1,2\}$

$$\begin{aligned}\mathbb{E}[X] &= p_X(0)\cdot 0 + p_X(1)\cdot 1 + p_X(2)\cdot 2 \\ &= \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 = 0 + \frac{1}{2} + \frac{1}{2} = 1.\end{aligned}$$

# Example 2

You roll a biased die.

It shows a 6 with probability $\frac{1}{3}$, and 1,…,5 with probability $2/15$ each.

Let $X$ be the value of the die. What is $\mathbb{E}[X]$?

# Example 2

You roll a biased die.

It shows a 6 with probability $\frac{1}{3}$, and 1,…,5 with probability $2/15$ each.

Let $X$ be the value of the die. What is $\mathbb{E}[X]$?

1. Write the PMF for $X$: $\mathrm{p}_X(k) = \begin{cases} 2/15 & k \in \{1,2,3,4,5\} \\ 1/3 & k = 6 \\ 0 & \text{otherwise} \end{cases}$

2. Plug in formula: $\mathbb{E}[X] = \frac{2}{15} \cdot 1 + \frac{2}{15} \cdot 2 + \frac{2}{15} \cdot 3 + \frac{2}{15} \cdot 4 + \frac{2}{15} \cdot 5 + \frac{1}{3} \cdot 6 = 4$

$\mathbb{E}[X]$ is not just the most likely outcome!

# Try it yourself

> Let $X$ be the result shown on a fair die. What is $\mathbb{E}[X]$?

> Let $Y$ be the sum of two (independent) fair die rolls. What is $\mathbb{E}[Y]$?

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# Try it yourself

Let $X$ be the result shown on a fair die. What is $\mathbb{E}[X]$

1. Write the PMF: $p_X(k) = \begin{cases} 1/6 & k \in \{1,2,3,4,5,6\} \\ 0 & \text{otherwise} \end{cases}$

2. Plug into formula: $\mathbb{E}[X] = 6 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$

$$= \frac{21}{6} = 3.5$$

$\mathbb{E}[X]$ is not necessarily a possible outcome!

That's ok, it's an average!

# Try it yourself

Let $Y$ be the sum of two (independent) fair die rolls. What is $\mathbb{E}[Y]$?

$\mathbb{E}[Y] =$

$\frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \frac{4}{36} \cdot 5 + \frac{5}{36} \cdot 6 + \frac{6}{36} \cdot 7 + \frac{5}{36} \cdot 8 + \frac{4}{36} \cdot 9 + \frac{3}{36} \cdot 10 + \frac{2}{36} \cdot 11 + \frac{1}{36} \cdot 12$

$= 7$

$\mathbb{E}[Y] = 2\mathbb{E}[X]$. That's not a coincidence…we'll talk about why next time!

# Subtle but Important

$X$ is random. You don't know what it is (at least until you run the experiment).

$\mathbb{E}[X]$ is not random. It's a number.

You don't need to run the experiment to know what it is.

# Summary

Today we have talked about **random variables** which assign quantitative values to outcomes of a random experiment.

How to describe a random variable?

- **Range/Support:** $\mathbf{\Omega_X}$ is set of possible values $X$ can be
- **Probability Mass Function:** $p_X(k) = \mathbb{P}(X = k)$
- **Cumulative Distribution Function:** $F_X(k) = \mathbb{P}(X \leq k)$
- **Expectation**: A single value that is the weighted average of values on the support, weighted on the probabilities $X$ takes on each
- **Variance:** Coming soon!