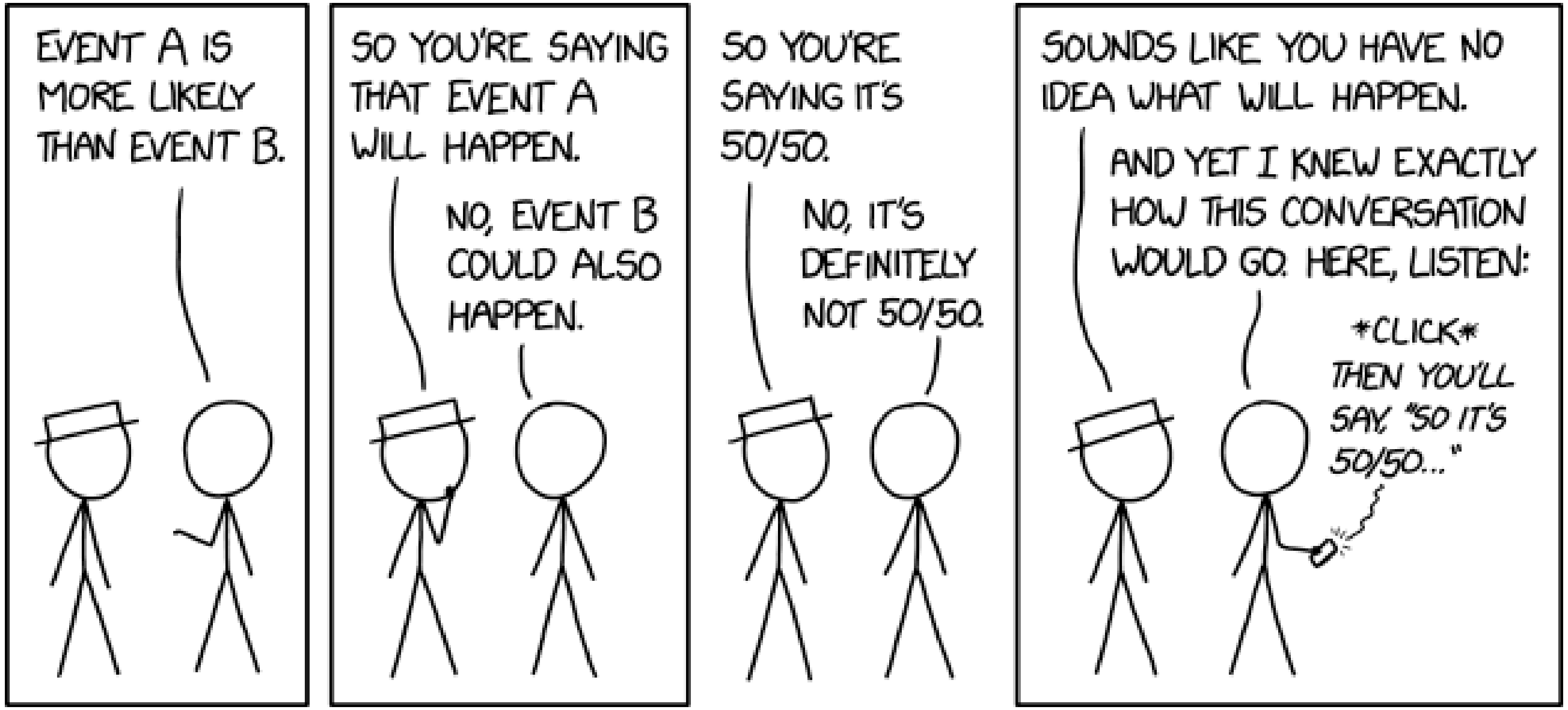

Probability

CSE 312 26Su

Lecture 4

Am I going too fast in lecture?
pollev.com/jolie312
(responses are anonymous)

https://xkcd.com/2370/

Announcements

- HW1 is due Wednesday (11:59pm)

Office hours schedule is on the calendar on the webpage

Where Are We?

Last time:

- Stars and Bars
- Combinatorial Proofs
- Pigeonhole Principle

Today:

- Probability
- Sample Space and Events
- Probability Axioms
- Uniform Probability Space
- More Examples

Today

So far…we've done a lot of counting.

Starting today, we get to calculate **probabilities**!

And the counting techniques we've learned are going to come in handy when computing probabilities ☺

Probability

What is **Probability**?

There are lots of things we aren’t certain about!

Is it going to rain this weekend?

Am I going to get a 6 when rolling this dice?

What is **Probability**?

There are lots of things we aren't certain about!

- Is it going to rain this weekend?
- Am I going to get a 6 when rolling this dice?

Probability is a way of <u>quantifying our uncertainty</u> when more than one outcome is possible

- What is the probability that is rains tomorrow?
- What is the probability that I get a 6 when rolling a dice?

What is **Probability**?

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Probability is a way of <u>quantifying our uncertainty</u> when more than one outcome is possible

What is the probability that is rains tomorrow?

What is the probability that I get a 6 when rolling a dice?

To have "real-world" examples, we're going to make some <u>assumptions</u>:

We can flip a coin, and each face is equally likely to come up

We can roll a die, and every number is equally likely to come up

We can shuffle a deck of cards so that every ordering is equally likely.

Experiment

Experiment

An action or process that leads to one or more outcomes.

A *random* experiment is an experiment where the outcome can't be predicted with certainty beforehand

Examples:

Tossing a fair coin

Rolling a dice

Drawing a name from a hat

Sample Space

Omega

Sample Space

A sample space Ω is the set of all possible outcomes of an experiment.

heads

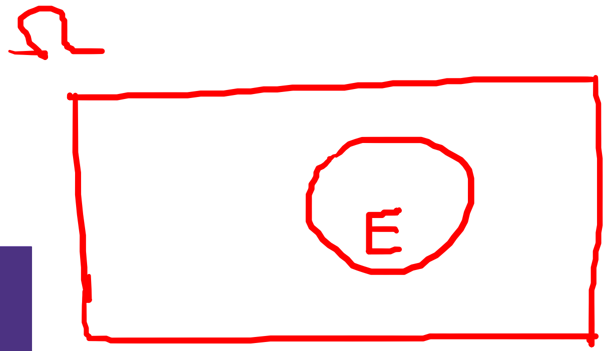
Examples:

For a single coin flip, $\Omega = \{H, T\}$

For a series of two coin flips, $\Omega = \{HH, HT, TH, TT\}$

For rolling a (normal) die: $\Omega = \{1,2,3,4,5,6\}$

Event



Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of the sample space Ω)

Examples:

Get a head in one coin flip ($E = \{H\}$)

Get at least one head among two coin flips ($E = \{HH, HT, TH\}$)

Get an even number on a die-roll ($E = \{2,4,6\}$).

Event

Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of the sample space Ω)

$H = \{H\}$

Examples:

Get a head in one coin flip ($E = \{H\}$)

$A = \{ \dots \}$

Get at least one head among two coin flips ($E = \{HH, HT, TH\}$)

Get an even number on a die-roll ($E = \{2,4,6\}$).

$N = \{ \dots$

Notation note: Since sets are usually represented with a single capital letter, we also denote events with a single capital letter (it doesn't have to be E, it can be anything!)

Examples

Experiment:
I roll a blue 4-sided die and a red 4-sided die.

The table contains the **sample space.**
$\Omega = \{(1,1), (1,2), \ldots\}$

blue ↓ \ red →	D2=1	D2=2	D2=3	D2=4
D1=1	(1,1)	(1,2)	(1,3)	(1,4)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Examples

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D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Let A be the event **that "the sum of the dice is even"**. The outcomes in this event are in gold

$A = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4)\}$

Examples

Experiment:
I roll a blue 4-sided die and a red 4-sided die.

The table contains the sample space.
$\Omega = \{(1,1),(1,2),\dots\}$

	D2=1	D2=2	D2=3	D2=4
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D1=2	(2,1)	(2,2)	(2,3)	(2,4)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Let A be the event that **"the sum of the dice is even"**. The outcomes in this event are in gold

$A = \{(1,1),(1,3),(2,2),(2,4),(3,1),(3,3),(4,2),(4,4)\}$

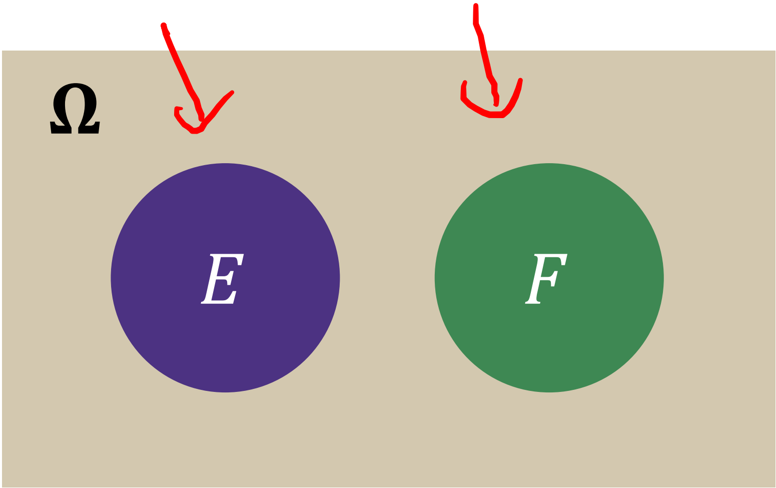
Let B be the event that **"first die is a 1"**. The outcomes in this event are in green.

$B = \{(1,1),(1,2),(1,3),(1,4)\}$

Mutually Exclusive Events

Two events E, F are **mutually exclusive if they can't happen simultaneously**.

In notation, $E \cap F = \emptyset$ (i.e. they're disjoint subsets of the sample space).



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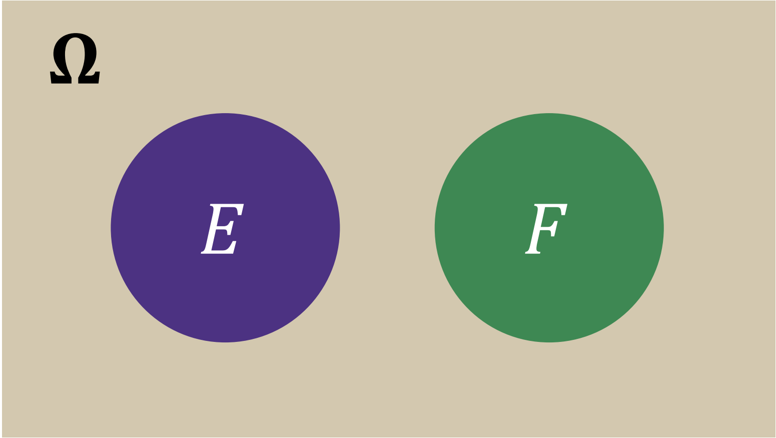
For example, if we flip a coin **and** roll a dice: $\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

E_1 ="the coin came up heads"

E_2 ="the coin came up tails"

E_3 ="the die showed an even number"



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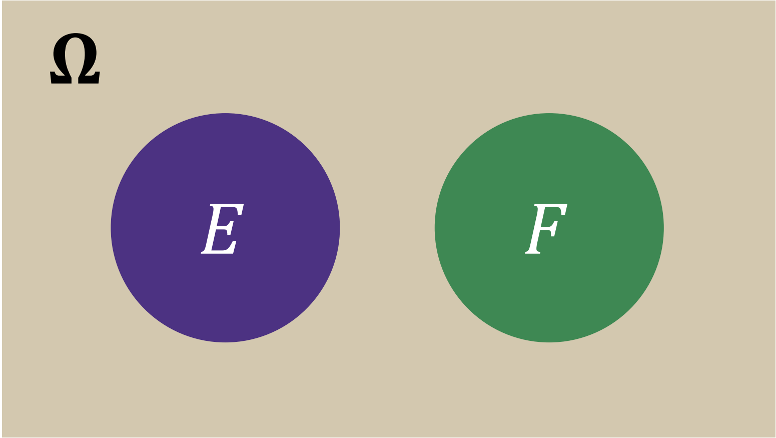
E_2 ="the coin came up tails"

E_3 ="the die showed an even number"



Are E_1 and E_2 mutually exclusive?
Are E_1 and E_3 mutually exclusive?

Mutually Exclusive Events

Two events E, F are **mutually exclusive if they can't happen simultaneously**.

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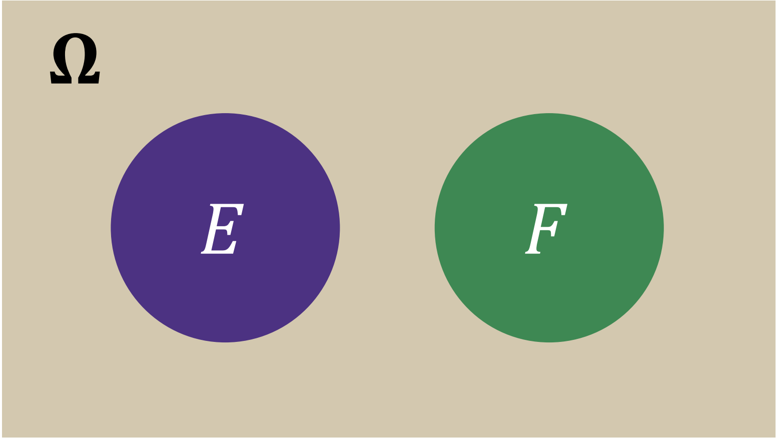


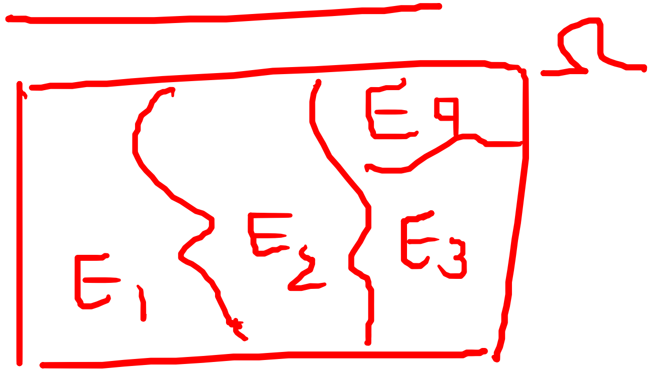
E_1 and E_2 are mutually exclusive.
E_1 and E_3 are not mutually exclusive.

Mutually Exclusive Events & Partitions

If $E_1, E_2, \ldots, E_n$ are events such that
- $E_i \cap E_j = \emptyset, \quad \forall i \neq j$
- and such that $\bigcup_{i=1}^{n} E_i = F$

then we say that events $E_1, E_2, \ldots, E_n$ **partition** set F.

$\bigcup_{i=1}^{n} E_i = \Omega$



Let events be
A – even rolls of red die
B – odd rolls of red die

$\Sigma = X$

$\Sigma \neq X$

Then A, B form a partition of Ω

What other examples can you think of?

	D2=1	D2=2	D2=3	D2=4
D1=1	(1,1)	(1,2)	(1,3)	(1,4)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Probability

Probability

A probability is a number between 0 and 1 describing how likely a particular outcome or event is.

omega

We define a function $\mathbb{P}$ that assigns a probability to every outcome ω in the sample space.

ω
(a particular outcome - an element of Ω)

→ $\mathbb{P}$ →

$\mathbb{P}(\omega)$ - A value in $[0,1]$
(the probability of that outcome)

mathcal?

Notation: $\mathbb{P}(\omega)$, $P(\omega)$, $\Pr(\omega)$ are all equivalent!

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

Recall: $\mathbb{P}$ assigns a probability to each outcome in the sample space

$P(H) = P(T) = \frac{1}{2}$

$P(H) = \frac{1}{4}$

$P(T) = \frac{3}{4}$

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

Recall: $\mathbb{P}$ assigns a probability to each outcome in the sample space

It depends on what we want to model!

If we have a *fair coin* $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$.

But we also might have a *biased coin*: $\mathbb{P}(H) = .85$, $\mathbb{P}(T) = 0.15$.

Probability Space

Probability Space

A **(discrete) probability space** is a pair $(\Omega, \mathbb{P})$ where:

$\boldsymbol{\Omega}$ is the sample space

$\mathbb{P}: \Omega \to [0,1]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(x) \geq 0$ for all x *(non-negativity)*
- $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$ *(normalization)*

Probability Space

Experiment: Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\} = \{(H, 1), (H, 2) \dots, (T, 1), (T, 2), \dots\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Is $(\Omega, \mathbb{P})$ a valid probability space?

Probability Space

Experiment: Flip a fair coin and roll a fair (6-sided) die.

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$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Is $(\Omega, \mathbb{P})$ a valid probability space?

☑ $\mathbb{P}$ takes in elements of Ω and outputs numbers between 0 and 1

☑ $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 12 \cdot \underline{\frac{1}{12} = 1}$.

Probability Space

Experiment: Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\} = \{(H, 1), (H, 2) \dots, (T, 1), (T, 2), \dots\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

So what's the probability of seeing the die roll be 2?

Seeing 2 isn't an element of the sample space!

Probability of An Event

Formally, $\mathbb{P}$ takes in only single outcomes. But…we will use the same notation to define the probability of an event (set of outcomes)!

$$\mathbb{P}(E) = \sum_{\omega \in \mathrm{E}} \mathbb{P}(\omega)$$

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Example:

Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$, $\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

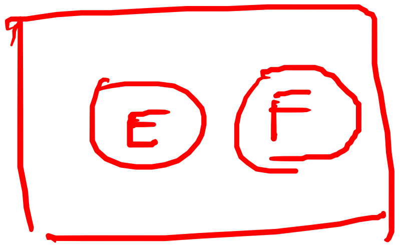
Let E be the event the dice is a 2. $E = \{(H, 2), (T, 2)\}$

$$\mathbb{P}(E) = \mathbb{P}\big((H, 2)\big) + \mathbb{P}\big((\underset{T}{H}, 2)\big) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

Probability Facts

Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events.



Axiom 1 (Non-negativity): $P(E) \geq 0$.
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Axioms of Probability & Corollaries

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events.

Axiom 1 (Non-negativity): $P(E) \geq 0.$
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If E and F are mutually exclusive,
then $P(E \cup F) = P(E) + P(F)$

Uniform Probability Spaces

A common type of probability space!

Uniform Probability Space

In a **uniform** probability space $(\Omega, \mathbb{P})$, every outcome in the sample space is **equally likely to occur**. For every outcome $\omega \in \Omega$,

$$\mathbb{P}(\omega) = \ ?$$

Uniform Probability Space

In a **uniform** probability space $(\Omega, \mathbb{P})$, every outcome in the sample space is **equally likely to occur**. For every outcome $\omega \in \Omega$,

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

For example,

> Flipping a <u>fair</u> coin: for every $\omega \in \Omega$, $\mathbb{P}(\omega) = \frac{1}{2}$

$\Omega = \{\cancel{1,2,3,4,}5,6\}$ H , T

> Rolling a <u>fair</u> dice: for every $\omega \in \Omega$, $\mathbb{P}(\omega) = \frac{1}{6}$

$\Omega = \{1,2,3,4,5,6\}$

Uniform Probability Space

In a **uniform** probability space $(\Omega, \mathbb{P})$, every outcome in the sample space is **equally likely to occur**. For every outcome $\omega \in \Omega$,

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

Finding the probability of an event in a uniform probability space:

$$\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega) = \sum_{\omega \in E} \frac{1}{|\Omega|} = \boxed{\frac{|E|}{|\Omega|}}$$

Uniform Probability Space (summarized ☺)

Uniform Probability Space

A **uniform probability space** is a probability space where all outcomes in the sample space are **equally likely** to occur.

> For every *outcome* $\omega \in \Omega$, $\mathbb{P}(\omega) = \frac{1}{|\Omega|}$

> For an *event* $E \subseteq \Omega$, $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$

Uniform Probability Space

HHHTHT - - -

TTTTT - - -

100

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

A. $\binom{100}{50}/2^{100}$

B. $1/101$

C. $1/2$

D. $1/2^{50}$

E. There is not enough information in this problem.

pollev.com/jolie312 and login with your UW net ID

Uniform Probability Space

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

E - "exactly 50 heads

$P(E) = \frac{|E|}{|\Omega|}$ _ $\underline{H}$ _ $\underline{H}$ _ $\underline{H}$ _   100

$|E| = \binom{100}{50}$

$|\Omega| = 2^{100}$

$= \frac{\binom{100}{50}}{2^{100}}$

Uniform Probability Space

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event “there are exactly 50 heads?

Ω is the set of all possible sequences of 100 coin tosses

$|\Omega| = 2^{100}$ because each of the 100 coin tosses have 2 options if it is head or tails

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Our probability measure is $\mathbb{P}(\omega) = \frac{1}{2^{100}}$ for every $\omega \in \Omega$

Uniform Probability Space

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

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$|\Omega| = 2^{100}$ because each of the 100 coin tosses have 2 options if it is head or tails

Our probability measure is $\mathbb{P}(\omega) = \frac{1}{2^{100}}$ for every $\omega \in \Omega$

Let H be the event that there are exactly 50 heads

$|H| = \binom{100}{50}$ because we pick which of the 50 coin tosses are heads – the rest are tails

$$\boldsymbol{P(H) = \frac{|H|}{|\Omega|} = \frac{\binom{100}{50}}{2^{100}}}$$

Uniform Probability Space

$2^{100} = |\Omega|$

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

⚠ We need to be careful how we define the sample space! ⚠

For example, if we defined the sample space as the number of heads in the sequence: $\Omega = \{1,2,\ldots,99,100\}$, we can't use a uniform probability space because every outcome is not equally likely here.

If we want to use a uniform probability, pick a sample space where every outcome is equally likely!

More examples

Mainly focusing on uniform probability spaces

More Examples!

1-6 1-6

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is the **sample space**? $\Omega = \{1,2,3,4,5,6\}^2$

What is the **probability measure** $\mathbb{P}$? $P(\omega) = \frac{1}{|\Omega|} = \frac{1}{36}$

What is the **event**? $E = \{2,4,6\}^2$

What is the **probability**?

$$P(E) = \frac{|E|}{|\Omega|} = \frac{3 \cdot 3}{6 \cdot 6} = \frac{9}{36}$$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is the **sample space**? $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

What is the **probability measure** $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is the **event**? $\{2,4,6\} \times \{2,4,6\}$

What is the **probability**? $3^2/6^2$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What if we defined our sample space as the *unordered* pairs of the die?

What is the **sample space**?

$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(2,5),(2,6)$
$(3,3),(3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)\}$ $|S| \neq 36$

What is the **probability measure** $\mathbb{P}$?

$(2,2)$

$\mathbb{P}((x,y)) = 2/36$ if $x \neq y$, $\mathbb{P}(x,x) = 1/36$

What is the **event**? $\{(2,2),(4,4),(6,6),(2,4),(2,6),(4,6)\}$ $|E| \neq 9$

What is the **probability**? $3 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} = \frac{9}{36}$

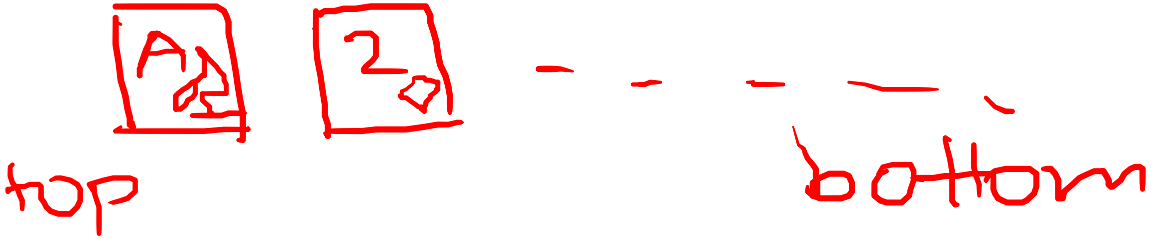
Takeaways

There is often more than one sample space possible! But one is probably easier than the others.

Finding a sample space that will make the uniform measure correct will usually make finding the probabilities easier to calculate.

- This often involves deciding what kind of information we need to encode in the sample space (e.g., should we care about order or not?)

Another Example



Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

$|\Omega| = 52!$ all 52 cards eq.

$\Omega = \{(x,y) : x, y \text{ are diff. cards}\}$

$|\Omega| = \frac{52!}{50!} = 52 \cdot 51$

Sample Space:

Probability Measure: $P(\omega) = \frac{1}{|\Omega|} = \frac{1}{52 \cdot 51}$

Event: 2 cards

Probability:

$$P(E) = \frac{|E|}{|\Omega|} = \frac{13 \cdot 4 \cdot 3}{52 \cdot 51}$$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: $\Omega = \{(x, y): x \text{ and } y \text{ are different cards } \}$

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$

Event: all pairs with equal values

Probability: $\frac{13 \cdot P(4,2)}{52 \cdot 51}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

$|\Omega| = 52!$

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50!}{52!}$ $|E|$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50 * 49 * 48 * \cdots . * 2 * 1}{52 * 51 * 50 * 49 * 48 * \ldots * 2 * 1}$

Takeaway

There’s often information you “don’t need” in your sample space.

It won’t give you the wrong answer.

But it sometimes makes for extra work/a harder counting problem,

Good indication: you cancelled A LOT of stuff that was common in the numerator and denominator.

Few notes about events and samples spaces

- If you're dealing with a situation where you may be able to use a uniform probability space, make sure to set up the sample space in a way that every outcome is equally likely.

- Try not overcomplicate the sample space – only include the information that you need in it.

- When you define an event, make sure it is a subset of the sample space!
  e.g., if order matters in the sample space, it should also matter in the event space

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if ?

$\mathbb{P}(E) = 1$ if and only if ?

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if an event can't happen.

$\mathbb{P}(E) = 1$ if and only if an event is guaranteed (every outcome outside E has probability 0).

Birthday Paradox

Sharing Birthdays

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

We know from the pigeonhole principle that if there are >365 people in the group, there will certainly be at least 2 people that share the same birthday. But what's the *probability* of this happening if there are only 50 people?

Sharing Birthdays🎂

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

What do you think this probability is closest to?

A) 0.001

B) 0.5

C) 0.99

D) 1

Let's do another poll!
pollev.com/cse312

Sharing Birthdays 🎂

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

Sample Space:

Probability Measure:

Event:

Probability:

Sharing Birthdays🎂

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

$\mathbb{P}$ (at least 2 people out of 50 people share a birthday)

= 1 - $\mathbb{P}$ (no one out of 50 people share a birthday)

What's our sample space?

$\mathbb{P}$ (no one out of 50 people share a birthday)

Sample space is the set of possible birthdays that 50 people can have.

- Birthdays can be repeated.

$|\Omega| = \text{number of possible birthday assignments} = 365^{50}$

What's our sample space and event?

$\mathbb{P}$ (no one out of 50 people share a birthday)

Event is set of birthday assignments such that no one shares a birthday.

$$|E| = P(365, 50) = \frac{365!}{(365-50)!}$$

$365 \cdot 364 \cdot 363 \cdot \ldots$

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

Sample Space: Set of assignments of birthdays to people. $|\Omega| = 365^{50}$

Probability Measure: Uniform probability measure. $\mathbb{P}(\omega) = \frac{1}{365^{50}}$ for $\omega \in \Omega$

Event: Let E be the event that at least 2 people share a birthday.

Probability: $\mathbb{P}(E) = 1 - \mathbb{P}(\bar{E})$. $\bar{E}$ is the event that **no one** shares a birthday.

$$\mathbb{P}(\bar{E}) = \frac{|\bar{E}|}{|\Omega|} = \frac{P(365,50)}{365^{50}}.$$

$|E| = P(365, 50)$

We use a permutation for $|\bar{E}|$ because birthdays are "selected" without replacement, (all have different birthdays) and order matters (my birthday is different from your birthday, etc.)

$$\mathbb{P}(E) = 1 - \frac{P(365,50)}{365^{50}} \approx 0.97.$$

Sharing Birthdays 🎂

There are about 50 people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

$$\mathbb{P}(E) = 1 - \frac{P(365,50)}{365^{50}} \approx 0.97.$$

This is pretty high! So almost definitely, two of us here share the same birthday 🥳

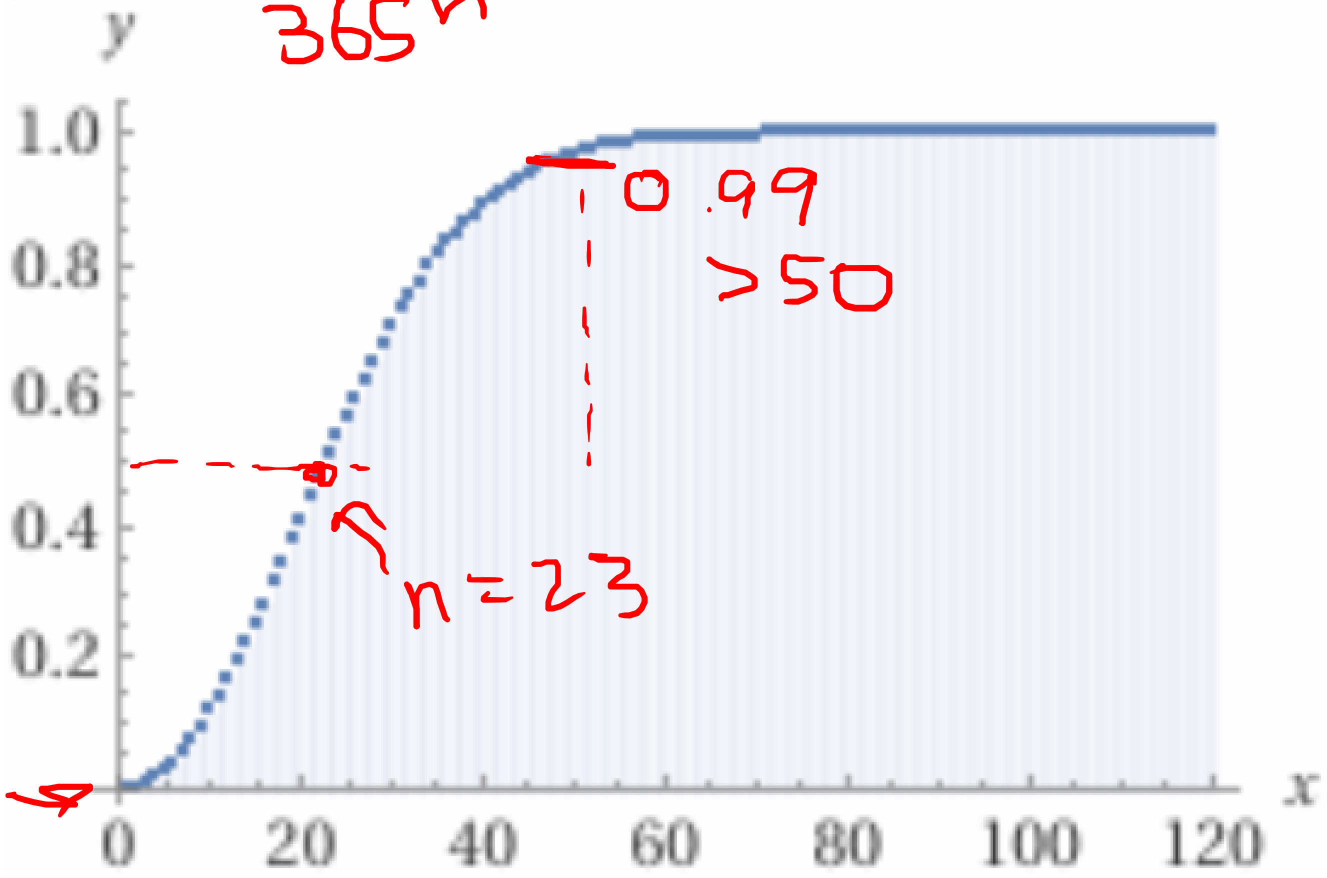
Sharing Birthdays Among n People 🎂

Suppose there are n people in this class. What is the probability that at least 2 of us share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

$$\mathbb{P}(E) = 1 - \frac{P(365, n)}{365^n}$$

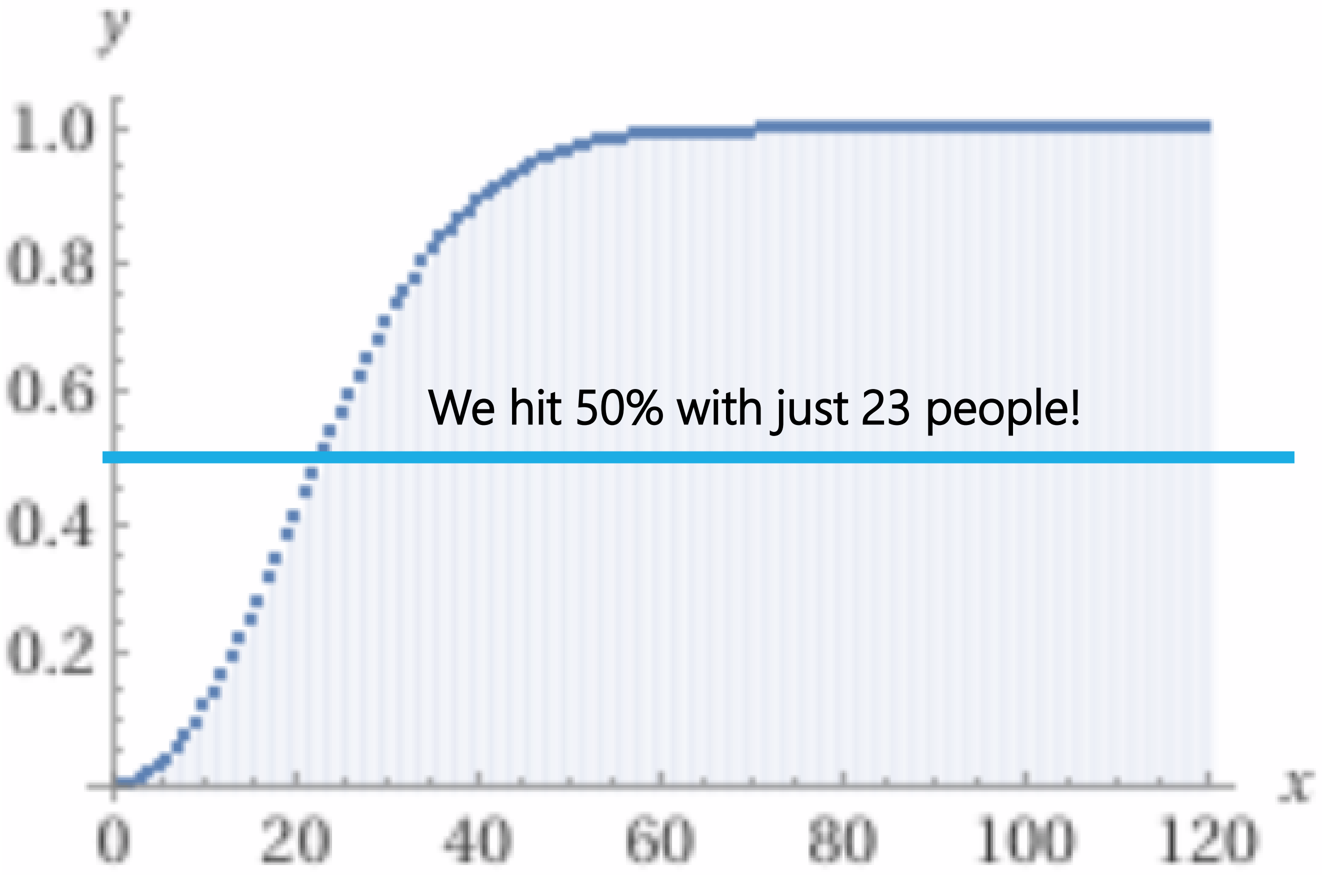
Sharing Birthdays Among n People 🎂

$$y = \frac{P(365, n)}{365^n}$$



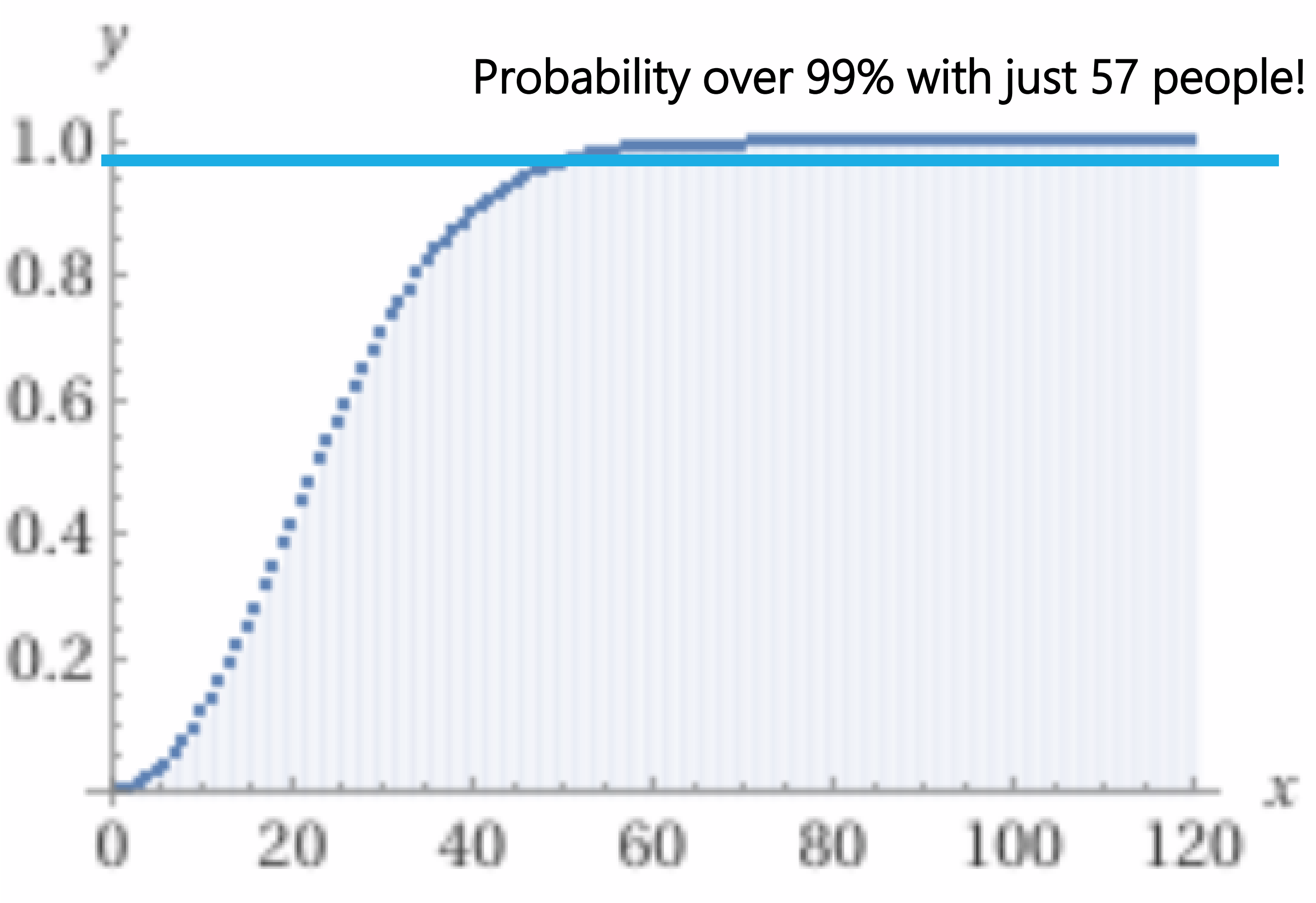
(x from 0 to 120)

Sharing Birthdays Among n People 🎂



(x from 0 to 120)

Sharing Birthdays Among n People🎂



(x from 0 to 120)

That’s very likely! Why?

It turns out that human brains find thinking about probabilities difficult!

Our brains are a bit selfish! When it comes to the probability that someone shares our birthday, that would be $\boxed{\frac{1}{365}}$ – not quite so likely.

But if we’re looking at any pair’s birthday in a group of n people, there are $\binom{n}{2}$ pairs of people, which grows quadratically with n. So the probability of at least one pair of people sharing a birthday approaches 1 pretty fast!

Summary

outcomes

- Probability allows us to assign a value between 0 and 1 to outomces
- A *random experiment* is any process where the outcome is not known for certain

Ω

- The *sample space* of an experiment is the set of all possible outcomes
- An *event* is a subset of the sample space (some set of outcomes)
- The *probability space* is the pair $(\Omega, \mathbb{P})$ where Ω is the sample space and $\mathbb{P}$ is the probability measure (a *function* that assigns probabilities to every outcome ω in the sample space)
  - A *uniform probability space* is a common type of probability space where every outcome is equally likely. To find the probability of an event in a uniform probability space, we find the size of the event divided by the size of the sample space

More Examples

Example 3

G R

Suppose I had a two, fair, 6-sided dice that we roll, one green, one red. What is the probability that we see at least one 3 in the two rolls?

Ω - all combos of G, R dice $= \{1,2,3,4,5,6\}^2$

$|\Omega| = 36$

Sample Space: $\to P(\omega) = \frac{1}{|\Omega|} = \frac{1}{36} \quad \forall \omega \in \Omega$

Probability Measure:

Event: A - seeing ≥ 1 three in G, R rolls

Probability: $P(A) = 1 - P(\bar{A})$

$\bar{A}$ - neither of G, R is 3 $\quad \bar{A} = \{1,2,4,5,6\}^2$

$|\bar{A}| = 5 \cdot 5 = 25$

$P(A) = 1 - \frac{|\bar{A}|}{|\Omega|} = 1 - \frac{25}{36} = \frac{11}{36}$

Example 3

Suppose I had a two, fair, 6-sided dice that we roll, one green, one red. What is the probability that we see at least one 3 in the two rolls?

Sample Space: $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

$|\Omega|$ is $6 \cdot 6 = 36$ because each of the dice rolls have 6 options.

Probability Measure: $\mathbb{P}(\omega) = \frac{1}{6^2} = \frac{1}{36}$

Event: Let A be the event that we see at least one 3 in the two rolls

Probability: $\mathbb{P}(A) = 1 - \mathbb{P}(\bar{A})$. $\bar{A}$ is the event that neither of the two rolls is a 3.$|\bar{A}| = 5^2 = 25$ because each roll has 5 options. $\mathbb{P}(\bar{A}) = \frac{|\bar{A}|}{|\Omega|} = \frac{25}{36}$.

So, $\mathbb{P}(A) = 1 - \frac{25}{36} = \frac{15}{36}$