

Random variables: Expectation and Linearity of Expectation

CSE 312 Spring 26
Lecture 9

Where we are

Last Class:

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)

Today:

- Recap
- Expectation
- Linearity of Expectation
- Indicator Random Variables

Kandinsky

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Review Random Variables

Definition. A **random variable (RV)** for a probability space (Ω, P) is a function $X: \Omega \rightarrow \mathbb{R}$.

The set of values that X can take on is its *range/support*: Ω_X

Review Random Variables and Associated Events

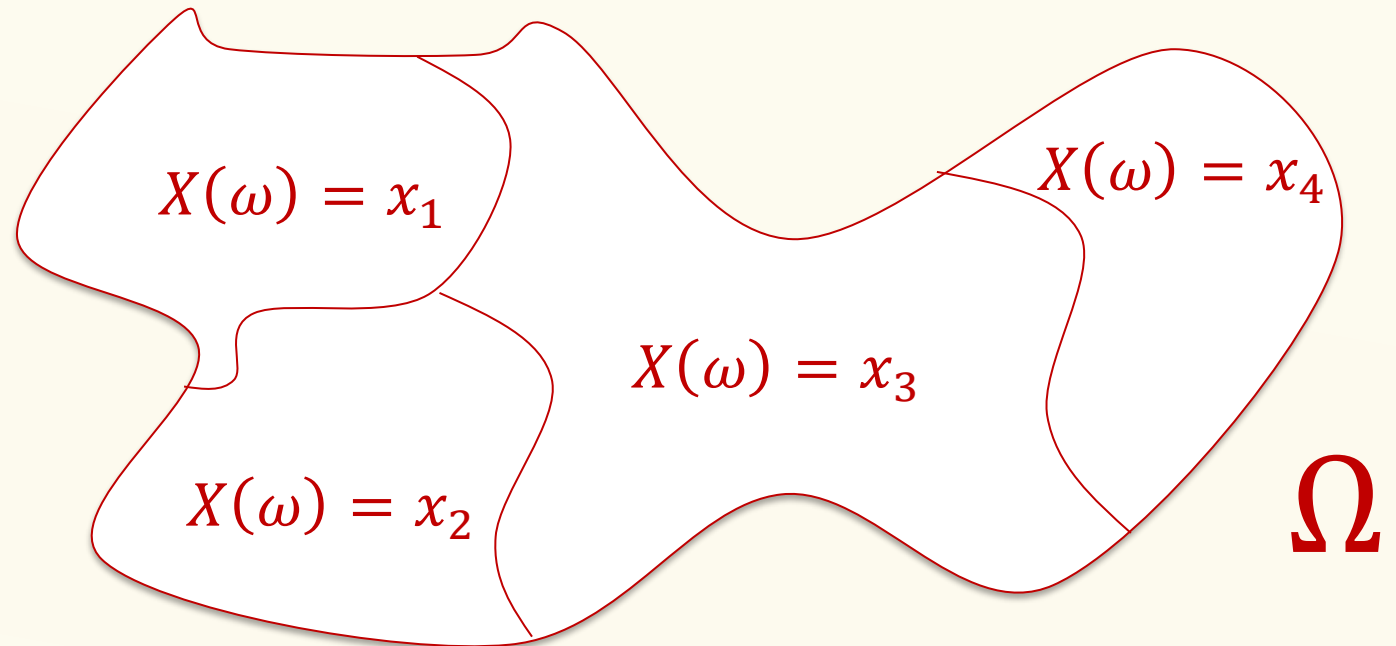
Definition. A **random variable (RV)** for a probability space (Ω, P) is a function $X: \Omega \rightarrow \mathbb{R}$.

The set of values that X can take on is its *range/support*: Ω_X

$$\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$$

Random variables **partition** the sample space.

$$\sum_{x \in \Omega_X} P(X = x) = 1$$



Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

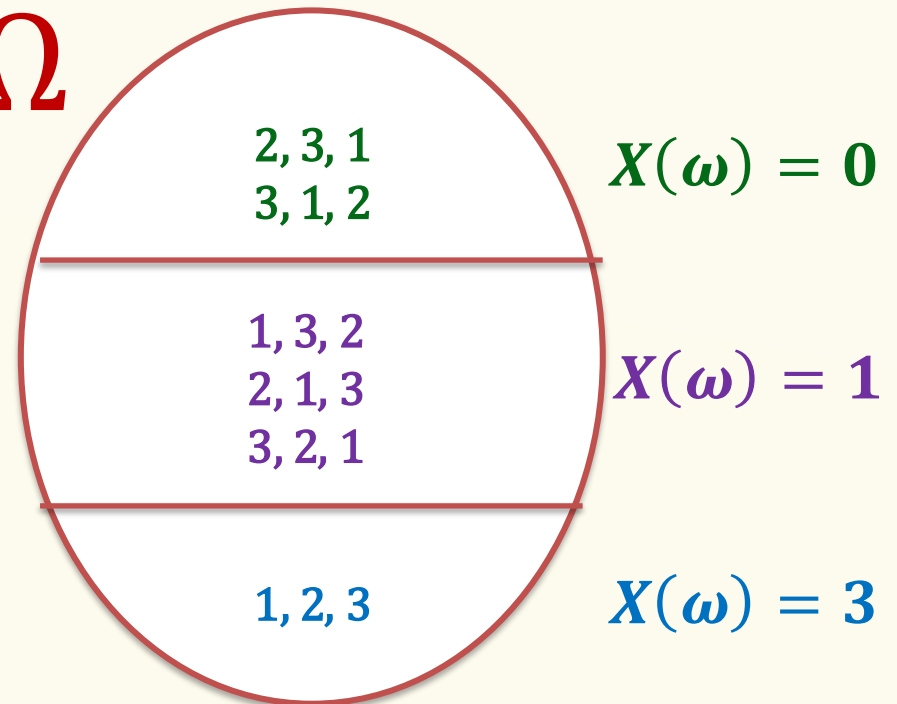
$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

$$P(X = 0) = 1/3$$

$$P(X = 1) = 1/2$$

$$P(X = 3) = 1/6$$

Ω



Review PMF and CDF

Definitions:

For a RV $X: \Omega \rightarrow \mathbb{R}$, the **probability mass function (pmf)** of X specifies, for any real number x , the probability that $X = x$

$$p_X(x) = P(X = x) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\sum_{x \in \Omega_X} p_X(x) = 1$$

For a RV $X: \Omega \rightarrow \mathbb{R}$, the **cumulative distribution function (cdf)** of X specifies, for any real number x , the probability that $X \leq x$

$$F_X(x) = P(X \leq x)$$

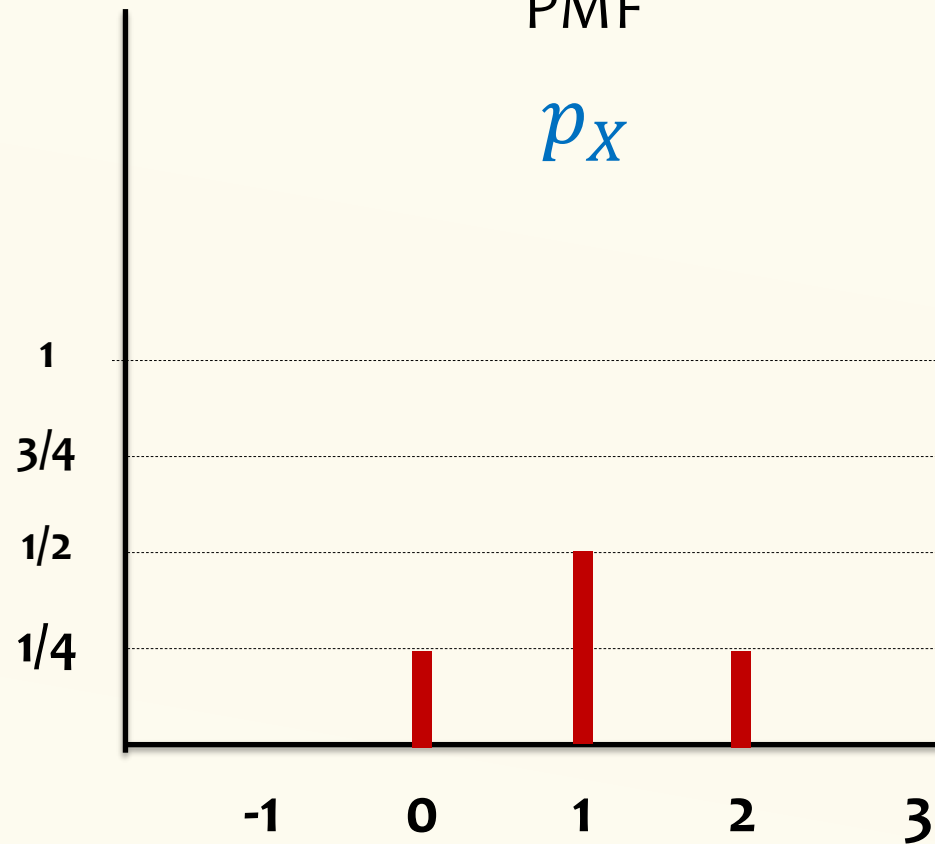
Example – Two fair independent coin flips

$X = \text{number of heads}$

Probability Mass Function

PMF

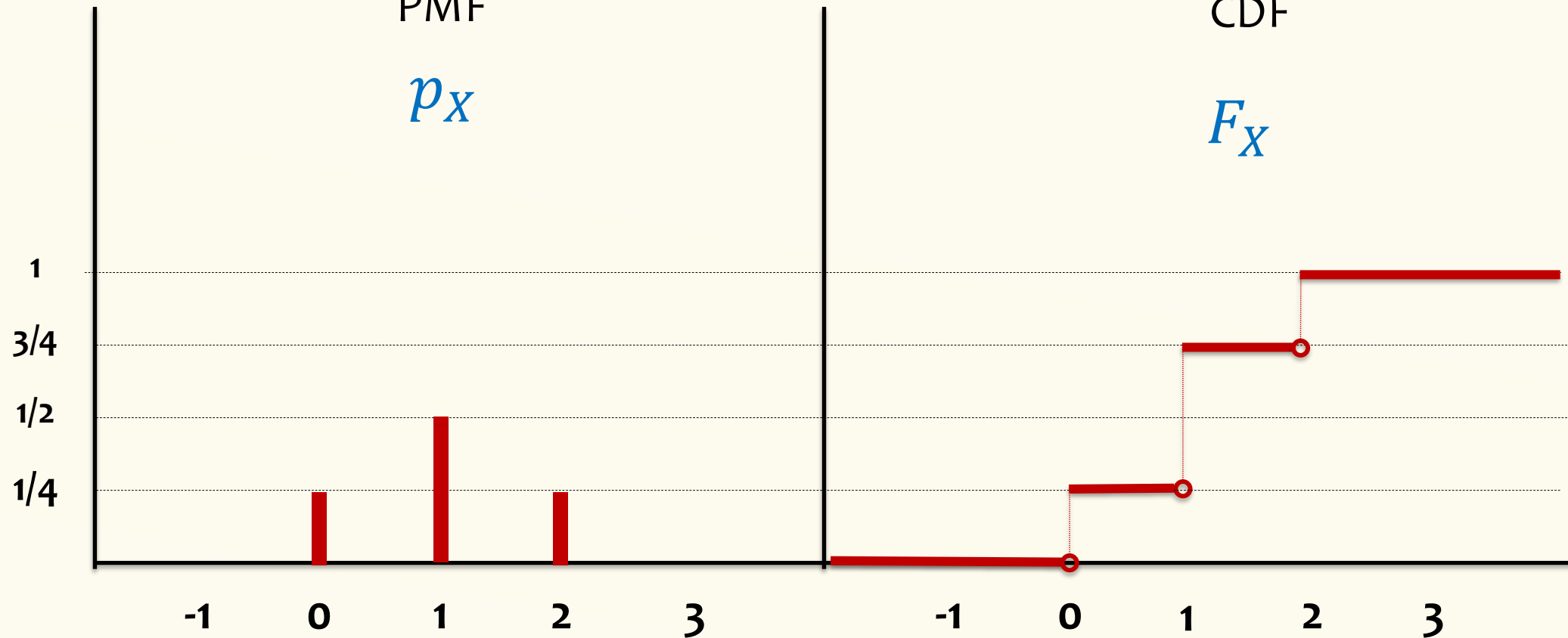
p_X



Cumulative Distribution Function

CDF

F_X



Example – Number of Heads - pmf

We flip n coins, independently, each heads with probability p

$$\Omega = \{\text{HH} \cdots \text{HH}, \text{HH} \cdots \text{HT}, \text{HH} \cdots \text{TH}, \dots, \text{TT} \cdots \text{TT}\}$$

$X = \#$ of heads

$$p_X(k) = P(X = k) = \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k}$$

of sequences with k heads

Prob of sequence w/ k heads

Agenda

- Expectation
- Linearity of Expectation
- Indicator Random Variables

Expectation (Idea)

Example. Toss a coin 20 times independently with probability $\frac{1}{4}$ of coming up heads on each toss.

X = number of heads

How many heads do you *expect* to see?

What if you toss it independently n times and it comes up heads with probability p each time?

Expected Value of a Random Variable

Definition. Given a discrete RV $X: \Omega \rightarrow \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

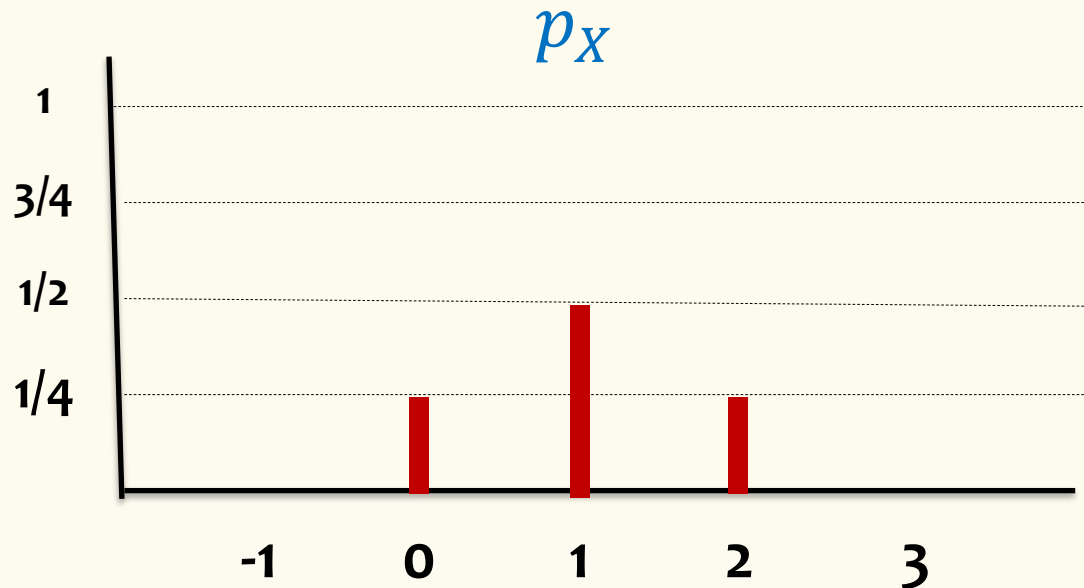
or equivalently

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x) = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: “Weighted average” of the possible outcomes (weighted by probability)

Expectation

Example. Two fair coin flips
 $\Omega = \{TT, HT, TH, HH\}$
 $X =$ number of heads



$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

What is $\mathbb{E}[X]$?

$$\begin{aligned} \mathbb{E}[X] &= X(TT) P(TT) + X(HT) P(HT) \\ &\quad + X(TH) P(TH) + X(HH) P(HH) \end{aligned}$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = 1$$

$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

Example: Returning Homeworks

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW
- What is $\mathbb{E}[X]$?

$\Pr(\omega)$	ω	$X(\omega)$
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	1, 3, 2	1
1/6	3, 2, 1	1
1/6	2, 1, 3	1
1/6	1, 2, 3	3

Returning Homeworks - Expectation

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

$$\begin{aligned}\mathbb{E}[X] &= 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} \\ &= 6 \cdot \frac{1}{6} = 1\end{aligned}$$

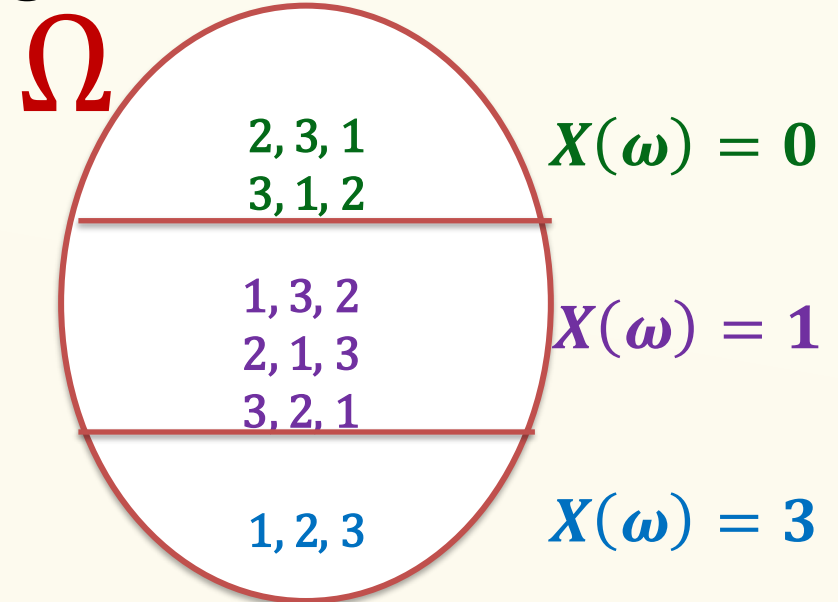
Returning Homeworks – Expectation other way

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1



$$\mathbb{E}[X] = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 3 \cdot P(X = 3) = 0 \cdot \frac{2}{6} + 1 \cdot \frac{3}{6} + 3 \cdot \frac{1}{6} = 1$$

Example – Coin Tosses

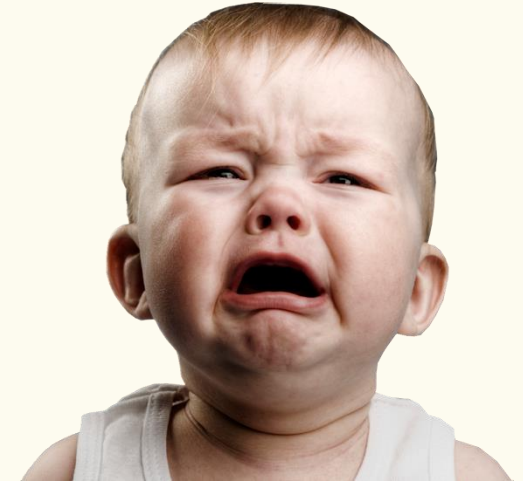
We flip n coins, each toss independent, probability p of coming up heads.

Z is the number of heads, what is $\mathbb{E}(Z)$?

Coin Tosses – Computing Expectation

We flip n coins, each toss independent; heads with probability p ,
 Z is the number of heads, what is $\mathbb{E}[Z]$?

$$\begin{aligned}\mathbb{E}[Z] &= \sum_{k=0}^n k \cdot P(Z = k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n k \cdot \frac{n!}{k! (n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^n \frac{n!}{(k-1)! (n-k)!} p^k (1-p)^{n-k} \\ &= np \sum_{k=1}^n \frac{(n-1)!}{(k-1)! (n-k)!} p^{k-1} (1-p)^{n-k} \\ &= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k! (n-1-k)!} p^k (1-p)^{(n-1)-k} \\ &= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np(p + (1-p))^{n-1} = np \cdot 1 = np\end{aligned}$$



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Can we solve it more elegantly, please?

Agenda (2)

- Expectation
- **Linearity of Expectation**
- Indicator Random Variables

Linearity of Expectation

Theorem. For **any** two random variables X and Y
(no conditions whatsoever on the random variables)

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Or, more generally: For any random variables X_1, \dots, X_n ,

$$\mathbb{E}[X_1 + \dots + X_n] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n].$$

Because: $\mathbb{E}[X_1 + \dots + X_n] = \mathbb{E}[(X_1 + \dots + X_{n-1}) + X_n]$
 $= \mathbb{E}[X_1 + \dots + X_{n-1}] + \mathbb{E}[X_n] = \dots$

Linearity of Expectation – Proof

Theorem. For **any** two random variables X and Y
(X, Y do not need to be independent)

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

$$\begin{aligned}\mathbb{E}[X + Y] &= \sum_{\omega} P(\omega)(X(\omega) + Y(\omega)) \\ &= \sum_{\omega} P(\omega)X(\omega) + \sum_{\omega} P(\omega)Y(\omega) \\ &= \mathbb{E}[X] + \mathbb{E}[Y]\end{aligned}$$

Using LOE to compute complicated expectations

Often boils down to the following three steps:

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

- Conquer: Compute the expectation of each X_i

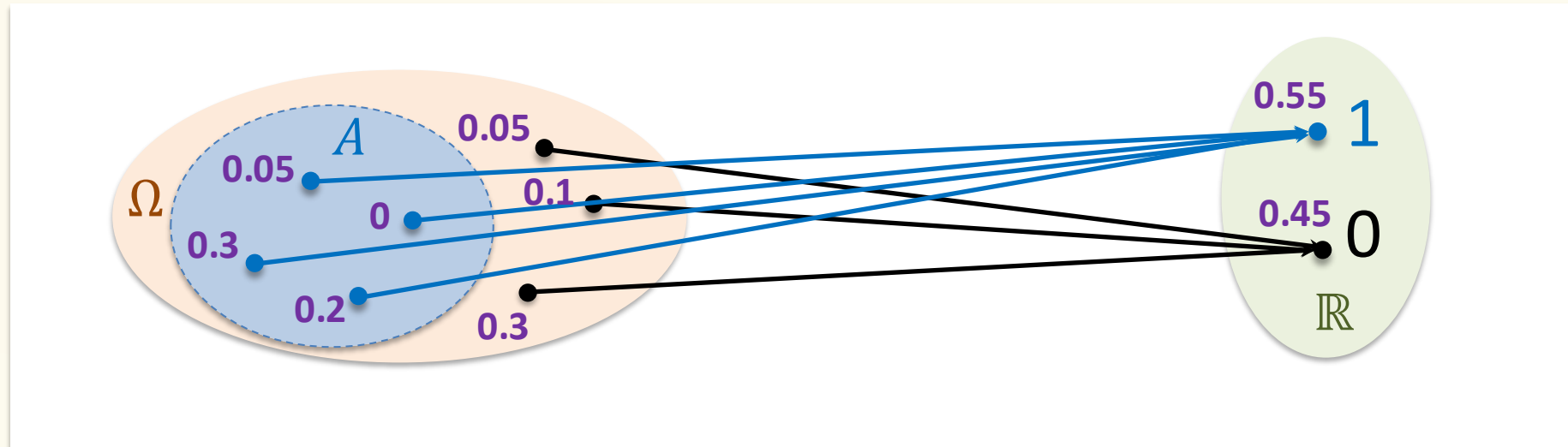
Often, X_i are **indicator** (0/1) random variables.

Indicator random variables – 0/1 valued

For any event A , can define the **indicator** random variable X_A for A

$$X_A = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{if event } A \text{ does not occur} \end{cases}$$

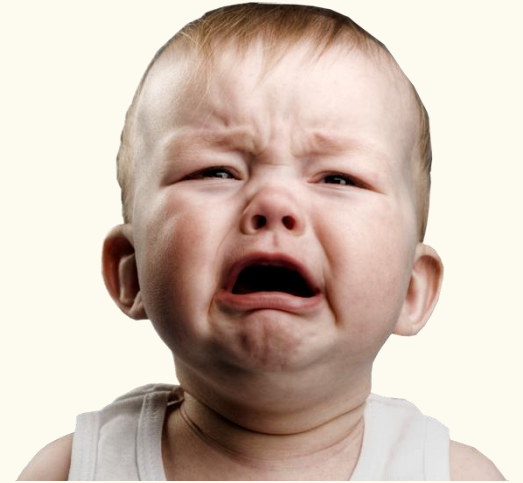
$$\begin{aligned} P(X_A = 1) &= P(A) \\ P(X_A = 0) &= 1 - P(A) \end{aligned}$$



Back to our example

We flip n coins, each toss independent; heads with probability p ,
 Z is the number of heads, what is $\mathbb{E}[Z]$?

$$\begin{aligned}\mathbb{E}[Z] &= \sum_{k=0}^n k \cdot P(Z = k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^n k \cdot \frac{n!}{k! (n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^n \frac{n!}{(k-1)! (n-k)!} p^k (1-p)^{n-k} \\ &= np \sum_{k=1}^n \frac{(n-1)!}{(k-1)! (n-k)!} p^{k-1} (1-p)^{n-k} \\ &= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k! (n-1-k)!} p^k (1-p)^{(n-1)-k} \\ &= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np(p + (1-p))^{n-1} = np \cdot 1 = np\end{aligned}$$



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Can we solve it more elegantly, please?

Computing complicated expectations

Often boils down to the following three steps:

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

- Conquer: Compute the expectation of each X_i

Often, X_i are **indicator** (0/1) random variables.

Indicator Random Variables for Counting Heads

We flip n coins, each toss independent, comes up heads with probability p
 Z is the number of heads, what is $\mathbb{E}[Z]$?

$$X_i = \begin{cases} 1, & i^{\text{th}} \text{ coin flip is heads} \\ 0, & i^{\text{th}} \text{ coin flip is tails.} \end{cases}$$

Fact. $Z = X_1 + \dots + X_n$

Outcome	X_1	X_2	X_3	Z
TTT	0	0	0	0
TTH	0	0	1	1
THT	0	1	0	1
THH	0	1	1	2
HTT	1	0	0	1
HTH	1	0	1	2
HHT	1	1	0	2
HHH	1	1	1	3

Wrapping up calculation using LOE

We flip n coins, each toss independent, comes up heads with probability p
 Z is the number of heads, what is $\mathbb{E}[Z]$?

$$X_i = \begin{cases} 1, & i^{\text{th}} \text{ coin flip is heads} \\ 0, & i^{\text{th}} \text{ coin flip is tails.} \end{cases}$$

$$\text{Fact. } Z = X_1 + \cdots + X_n$$

Linearity of Expectation:

$$\mathbb{E}[Z] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n \cdot p$$

$$\begin{aligned} P(X_i = 1) &= p \\ P(X_i = 0) &= 1 - p \end{aligned}$$

$$\mathbb{E}[X_i] = p \cdot 1 + (1 - p) \cdot 0 = p$$

Returning Homeworks – more students

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$
$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

$$\mathbb{E}[X] = 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$$
$$= 6 \cdot \frac{1}{6} = 1$$

$$\mathbb{E}[X] = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 3 \cdot P(X = 3)$$
$$= 0 \cdot \frac{2}{6} + 1 \cdot \frac{3}{6} + 3 \cdot \frac{1}{6} = 1$$

- What if there are n students?

Returning Homeworks - n students

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

$$\mathbb{E}[X] = \sum_{k=0}^n k \cdot P(X = k)$$

What is $P(X = k)$?

- What if there are n students?

Returning Homeworks using LOE

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

Decompose: Find the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Conquer: Compute the expectation of each X_i and sum!

Returning Homeworks using LOE – indicator random variables

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$?

$\Pr(\omega)$	ω	$X(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$X_3(\omega)$
1/6	1, 2, 3	3	1	1	1
1/6	1, 3, 2	1	1	0	0
1/6	2, 1, 3	1	0	0	1
1/6	2, 3, 1	0	0	0	0
1/6	3, 1, 2	0	0	0	0
1/6	3, 2, 1	1	0	1	0

Returning Homeworks using LOE (2)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

Decompose:

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

LOE:

Conquer:

Returning Homeworks using LOE (3)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X_i]$?

$\Pr(\omega)$	ω	$X(\omega)$	$X_1(\omega)$	$X_2(\omega)$	$X_3(\omega)$
1/6	1, 2, 3	3	1	1	1
1/6	1, 3, 2	1	1	0	0
1/6	2, 1, 3	1	0	0	1
1/6	2, 3, 1	0	0	0	0
1/6	3, 1, 2	0	0	0	0
1/6	3, 2, 1	1	0	1	0

Returning Homeworks using LOE (4)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

Decompose: What is X_i ?

$X_i = 1$ iff i^{th} student gets own HW back; 0 o.w.

LOE: $\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$

Conquer: $\mathbb{E}[X_i] = \frac{1}{n}$

Therefore, $\mathbb{E}[X] = n \cdot \frac{1}{n} = 1$

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

Pairs with the same birthday

- In a class of m students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?
- Each person's birthday is equally likely to be any of the 365 possibilities and different people's bdays are independent.

Pairs with the same birthday - LOE

- In a class of m students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?
- Each person's birthday is equally likely to be any of the 365 possibilities and different people's bdays are independent.

Decompose: Find the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Conquer: Compute the expectation of each X_i and sum!

Pairs with the same birthday – LOE (2)

- In a class of m students, let X be the number of pairs of people with the same birthday (assuming 365 equally likely birthdays)?
- What is $\mathbb{E}[X]$?

Decompose: Indicator events involve **pairs** of students (i, j) for $i \neq j$
 $X_{ij} = 1$ iff students i and j have the same birthday

LOE: $\binom{m}{2}$ indicator variables X_{ij}

Conquer: $\mathbb{E}[X_{ij}] = \frac{1}{365}$ so total expectation is $\frac{\binom{m}{2}}{365}$ pairs

Linearity of Expectation – Even stronger

Theorem. For any random variables X_1, \dots, X_n , and real numbers $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}[a_1X_1 + \dots + a_nX_n] = a_1\mathbb{E}[X_1] + \dots + a_n\mathbb{E}[X_n].$$

Very important: In general, we do not have $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$