

More on Independence

CSE 312 Spring 26
Lecture 7

Review - Conditional Probability

Definition. The **conditional probability** of event E given an event F happened (assuming $P(F) \neq 0$) is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

If the probability space is uniform, then $P(E|F) = \frac{|E \cap F|}{|F|}$

Review - Chain Rule (general case)



$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} \quad \longrightarrow \quad \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{B})$$

Theorem. (Chain Rule) For events $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$,

$$\mathbb{P}(\mathcal{A}_1 \cap \dots \cap \mathcal{A}_n) = \mathbb{P}(\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_2|\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_3|\mathcal{A}_1 \cap \mathcal{A}_2)$$

$$\dots \mathbb{P}(\mathcal{A}_n|\mathcal{A}_1 \cap \mathcal{A}_2 \cap \dots \cap \mathcal{A}_{n-1})$$

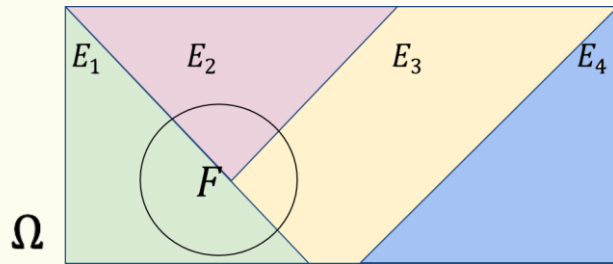
An easy way to remember: We have n tasks and we can do them **sequentially**, conditioning on the outcome of previous tasks

Review: Conditional & Total Probabilities

- **Conditional Probability**

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{if } P(A) \neq 0$$

- **Law of Total Probability**



E_1, \dots, E_n partition Ω

$$P(F) = \sum_{i=1}^n P(F \cap E_i) = \sum_{i=1}^n P(F|E_i)P(E_i)$$

- **Bayes Theorem**

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{if } P(A) \neq 0, P(B) \neq 0$$

Review: Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let E_1, E_2, \dots, E_n be a partition of the sample space, and F an event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^n P(F|E_i)P(E_i)}$$

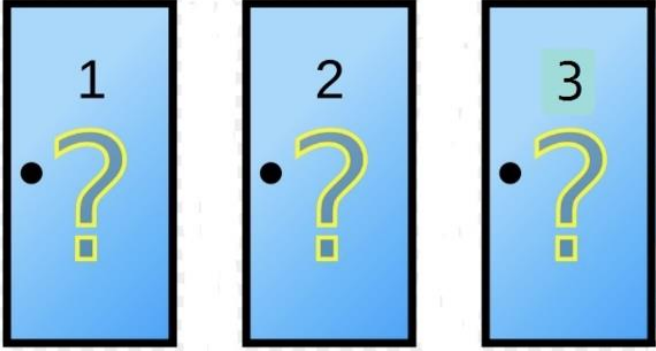
Simple Partition: In particular, if E is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^c)P(E^c)}$$

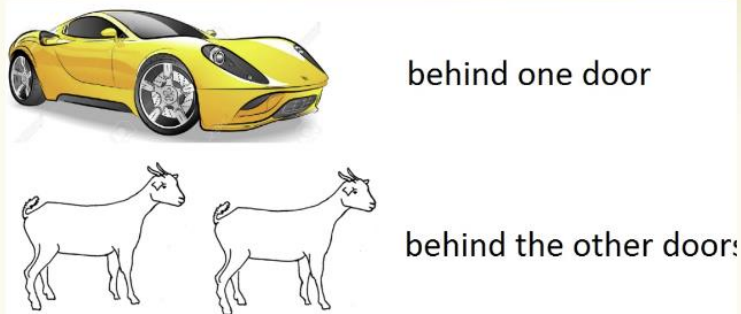
Agenda

- Recap
- The Monty Hall Problem
- Conditional Probability Defines Probability Space
- **More on independence**
 - Example using Law of Total Probability
 - Independence of many events
 - Independence not always obvious
 - Defining a probability space using independence
 - Independence as an assumption not always justified

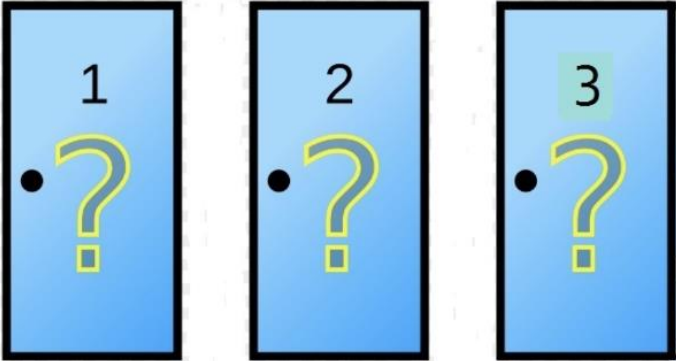
The Monty Hall Problem



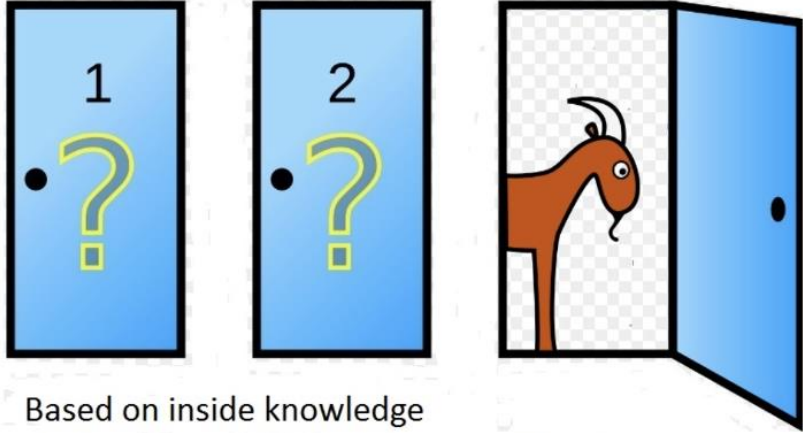
Suppose you're on a game show
You're given the choice of three doors



Car equally likely to be behind any of 3 doors



You pick a door, say No. 1
Say, pick uniformly at random



Based on inside knowledge
the host opens another door, say No. 3

If host has two options, picks uniformly at random.

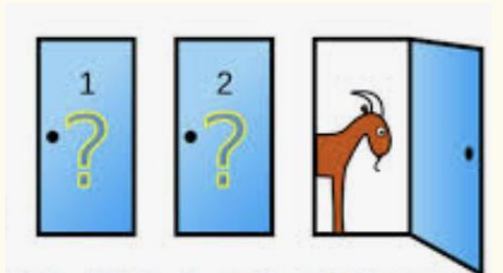
Finally, you are offered the option to switch to door 2. Should you?

In words....

Suppose you're on a game show, and you're given the choice of three doors. Behind one of the doors is a car, behind the other, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to switch to door number 2?" Is it to your advantage to switch your choice of doors?

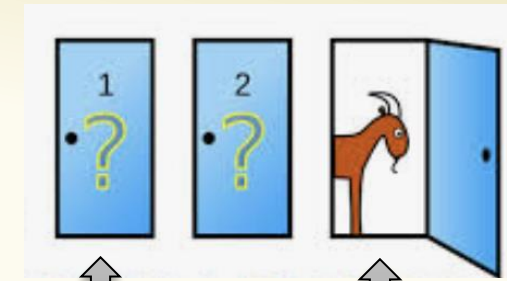
Assumptions

- The player is equally likely to pick each of the three doors, **say door 1**.
- After the player picks a door, the host **must** open a different door with a goat behind it and offer the player the choice of staying with the original door he selected or switching to the other unopened door
- If the host has two options, he picks one at random.



Should you stay or should you switch?

- P_i = the prize is behind door i.
- C_i = the contestant picks door i, say **door 1**.
- R_i = the host reveals door i, say **door 3**.



C_1 = Contestant picks door 1

R_3 = Host reveals door 3

$$\mathbb{P}(\text{prize behind door 2} \mid C_1 \cap R_3) = \frac{\mathbb{P}(P_2 \cap C_1 \cap R_3)}{\mathbb{P}(C_1 \cap R_3)}$$

$$\mathbb{P}(\text{prize behind door 1} \mid C_1 \cap R_3) = \frac{\mathbb{P}(P_1 \cap C_1 \cap R_3)}{\mathbb{P}(C_1 \cap R_3)}$$

Should you stay or should you switch (written out)?

- P_i = the prize is behind door i .
- C_i = the contestant picks door i , say **door 1**.
- R_i = the host reveals door i , say **door 3**.
- $\mathbb{P}(P_2 \cap C_1 \cap R_3) = \mathbb{P}(P_2)\mathbb{P}(C_1|P_2)\mathbb{P}(R_3|P_2 \cap C_1) = \frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}$
- $\mathbb{P}(P_1 \cap C_1 \cap R_3) = \mathbb{P}(P_1)\mathbb{P}(C_1|P_1)\mathbb{P}(R_3|P_1 \cap C_1) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18}$
- Since P_1, P_2, P_3 partition the sample space, by the law of total probability
- $\mathbb{P}(C_1 \cap R_3) = \mathbb{P}(P_1 \cap C_1 \cap R_3) + \mathbb{P}(P_2 \cap C_1 \cap R_3) + \mathbb{P}(P_3 \cap C_1 \cap R_3) = \frac{1}{18} + \frac{1}{9} + 0 = \frac{1}{6}$

$$\mathbb{P}(P_2 | C_1 \cap R_3) = \frac{\mathbb{P}(P_2 \cap C_1 \cap R_3)}{\mathbb{P}(C_1 \cap R_3)} = \frac{\frac{1}{9}}{\frac{1}{6}} = \frac{2}{3}$$

$$\mathbb{P}(P_1 | C_1 \cap R_3) = \frac{\mathbb{P}(P_1 \cap C_1 \cap R_3)}{\mathbb{P}(C_1 \cap R_3)} = \frac{\frac{1}{18}}{\frac{1}{6}} = \frac{1}{3}$$

Hence, the contestant would be better off switching to door 3!!

Agenda (2)

- Recap
- The Monty Hall Problem
- **Conditional Probability Defines Probability Space**
- More on independence

Zika Testing – different question

Z = you have Zika

T = you test positive for Zika

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika (“true positive”)
- However, the test may yield a “false positive” 1% of the time
- 0.5% of the US population has Zika.

$$P(T | Z) = 0.98$$

$$P(T | \bar{Z}) = 0.01$$

$$P(Z) = 0.005$$

What is the probability you test negative (event \bar{T}) if you have Zika (event Z)?

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$\Pr(\bar{T} | Z)$

A) 0.02

B) 0.99

C) 0.02×0.005

D) 0.02×0.995

Conditional Probability Define a Probability Space

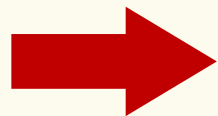
Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Conditional Probability Define a Probability Space (2)

The probability conditioned on A follows the same properties as (unconditional) probability.

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Formally. (Ω, \mathbb{P}) is a probability space + $\mathbb{P}(\mathcal{A}) > 0$



$(\mathcal{A}, \mathbb{P}(\cdot | \mathcal{A}))$ is a probability space

Axiom 1 (Non-negativity): $P(E) \geq 0$.

Axiom 2 (Normalization): $P(\Omega) = 1$

Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

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Independence

Definition. Two events \mathcal{A} and \mathcal{B} are **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

Alternatively,

- If $\mathbb{P}(\mathcal{A}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{B})$
- If $\mathbb{P}(\mathcal{B}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$

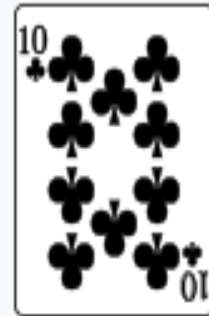
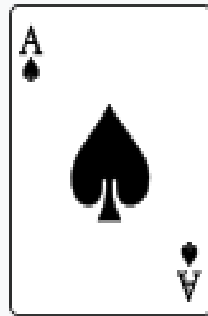
“The probability that \mathcal{B} occurs after observing \mathcal{A} ” -- Posterior
= “The probability that \mathcal{B} occurs” -- Prior

Are A and B independent?

Have a Standard 52-Card Deck. Shuffle It, and draw the top 2 cards **in order**:

Step 1: Draw a uniformly random card.

Step 2: Draw a uniformly random card from remaining cards.



A: Ace of Spades First

B: 10 of Clubs Second

$$\mathbb{P}(B|A) = \frac{1}{51}$$

$$\mathbb{P}(B) =$$

Are the events A and B independent?

Calculation

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Pr (2nd card = 10 clubs | 1st card = X)

- A) 1/52
- B) 1/51
- C) Depends on X
- D) (1/52) x (1/51)

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Multiple Events – Mutual Independence

Definition. Events A_1, \dots, A_n are **mutually independent** if for every non-empty subset $I \subseteq \{1, \dots, n\}$, we have

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i).$$

Multiple Events – Mutual Independence - corollary

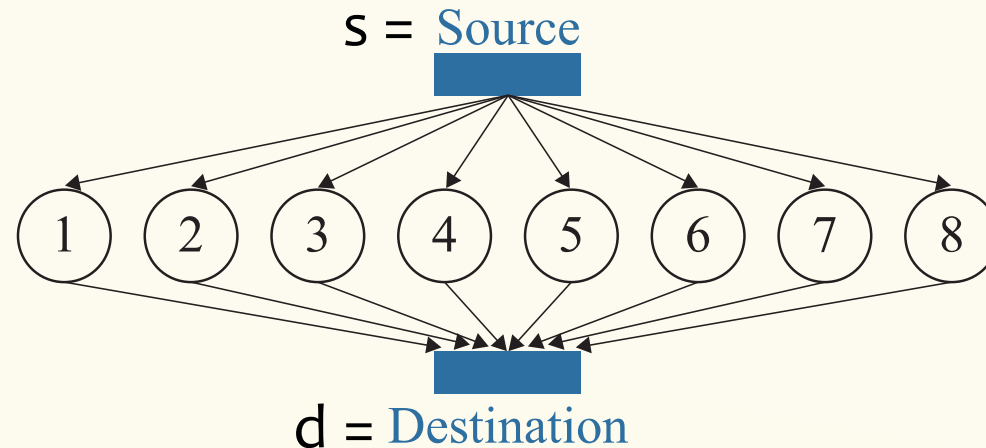
if A_1, \dots, A_n are **mutually independent** then for any choice of $B_i \in \{A_i, A_i^c\}$, $i = 1..n$ and every non-empty subset $I \subseteq \{1, \dots, n\}$, we have

$$P\left(\bigcap_{i \in I} B_i\right) = \prod_{i \in I} P(B_i).$$

An example

You are routing a packet from the source to the destination.

But each of the 16 edges in the network works with probability p , independently.

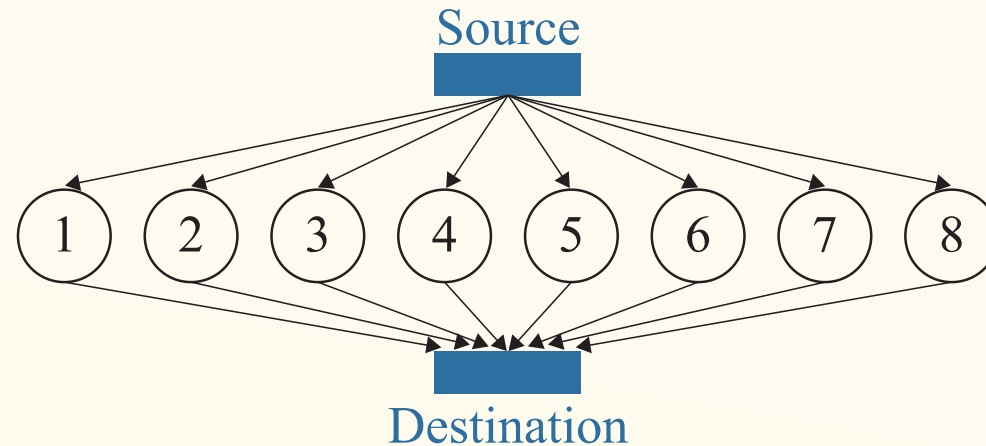


Q: $\Pr(\text{edge } (s,3) \text{ works, edge } (5,d) \text{ works, edge } (s,7) \text{ doesn't work})?$

An example (2)

You are routing a packet from the source to the destination.

But each of the 16 edges in the network works with probability p , independently.



Q: What is the probability that you can get the packet from the source to the destination?

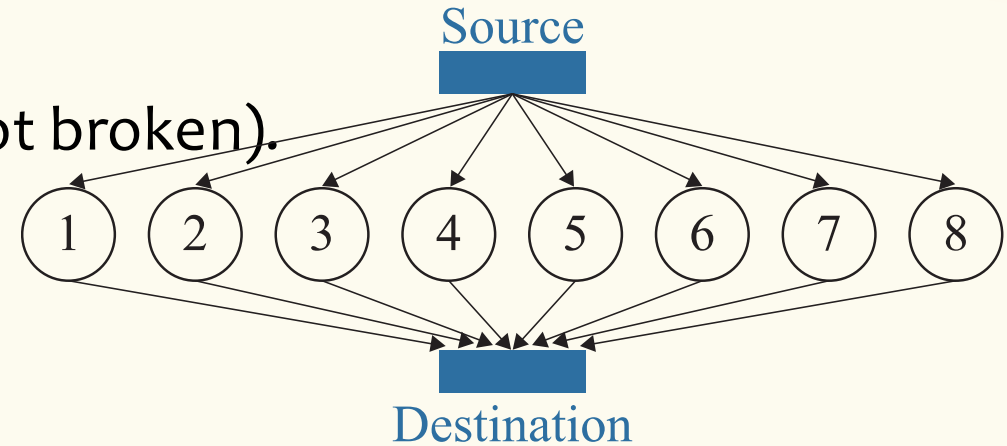
Let's break it down

Each edge works with probability p , independently of other edges. There are 8 paths.

Let E_i denote the event that the i^{th} path is usable (not broken).

Q1: What is $P\{E_i\}$?

Q2: What is $P\{\bar{E}_i\}$?



More calculations

Each edge works with probability p , independently of other edges. There are 8 paths.

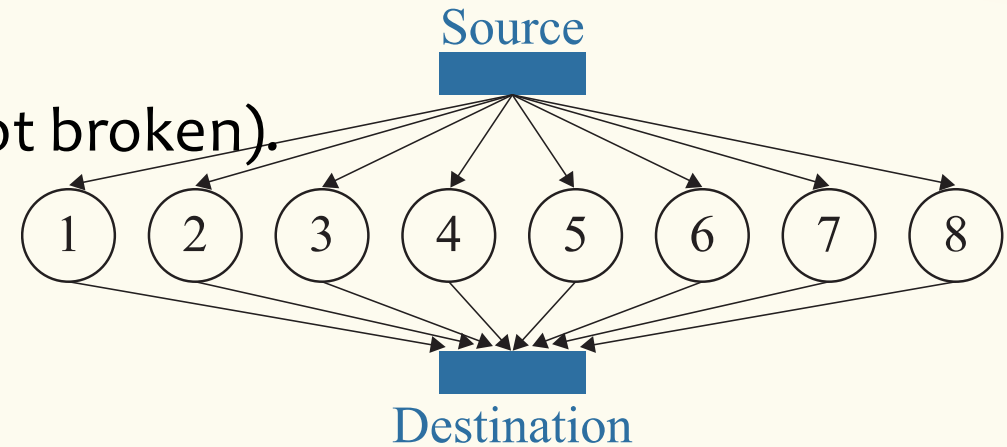
Let E_i denote the event that the i^{th} path is usable (not broken).

$$P\{E_i\} = p^2$$

$$P\{\bar{E}_i\} = 1 - p^2$$

What is $P\{\text{Can get from source to destination}\}$?

$P\{\text{Can get from source to destination}\} =$



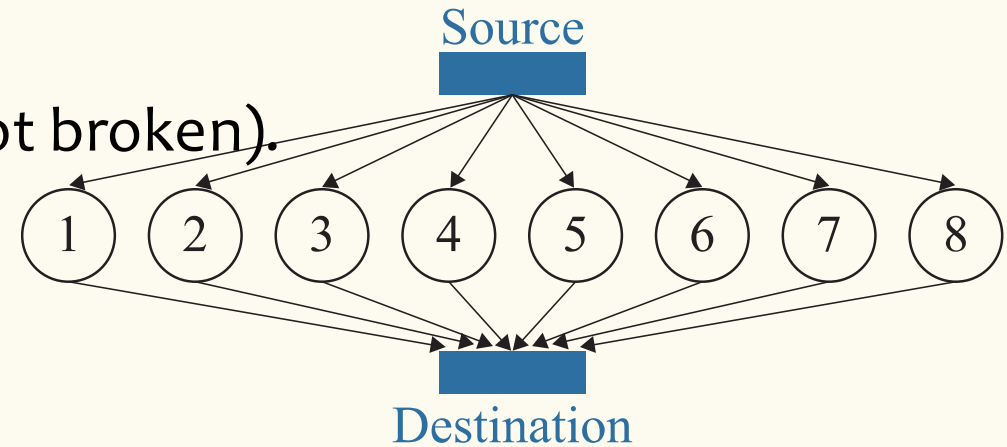
Answer

Each edge works with probability p , independently of other edges. There are 8 paths.

Let E_i denote the event that the i^{th} path is usable (not broken).

$$P\{E_i\} = p^2$$

$$P\{\bar{E}_i\} = 1 - p^2$$



What is $P\{\text{Can get from source to destination}\}$?

$$P\{\text{Can get from source to destination}\} = P\{\text{At least one path works}\}$$

$$= P\{E_1 \cup E_2 \cup \dots \cup E_8\}$$

$$= 1 - P\{\text{All paths are broken}\}$$

$$= 1 - P\{\bar{E}_1\} \cdot P\{\bar{E}_2\} \dots P\{\bar{E}_8\} = 1 - (1 - p^2)^8$$

Agenda (5)

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- **More on independence**
 - Example using Law of Total Probability
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 - **Defining a probability space using independence**

Defining a probability space

Often probability space (Ω, \mathbb{P}) is **defined** using independence

Example – Biased coin

We have a biased coin comes up Heads with probability $2/3$; Each flip is independent of all other flips. Suppose it is tossed 3 times.

$$\mathbb{P}(HHH) =$$

$$\mathbb{P}(TTT) =$$

$$\mathbb{P}(HTT) =$$

Example – Biased coin – let's do a calculation

We have a biased coin comes up Heads with probability $2/3$, independently of other flips. Suppose it is tossed 3 times.

$\mathbb{P}(2 \text{ heads in } 3 \text{ tosses}) =$

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- A) $(2/3)^2 1/3$
- B) $2/3$
- C) $3 (2/3)^2 1/3$
- D) $(1/3)^2$

Independence as an assumption

- People often assume it **without justification.**
- Example: A sky diver has two chutes

A : event that the main chute doesn't open

$$\mathbb{P}(A) = 0.02$$

B : event that the backup doesn't open

$$\mathbb{P}(B) = 0.1$$

- What is the chance that at least one opens assuming independence?

$$1 - \mathbb{P}(A) \cdot \mathbb{P}(B) = 1 - 0.02 \cdot 0.1 = 0.998$$

But assuming independence doesn't justify the assumption! Both chutes could fail because of the same rare event e.g., freezing rain.

Agenda (6)

- Recap
- The Monty Hall Problem
- Conditional Probability Defines Probability Space
- More on Independence
- **Conditional independence**

Conditional Independence

Definition. Two events A and B are **independent** conditioned on C if $P(C) \neq 0$ and $P(A \cap B | C) = P(A | C) \cdot P(B | C)$.

- If $P(A \cap C) \neq 0$, equivalent to $P(B|A \cap C) = P(B | C)$
- If $P(B \cap C) \neq 0$, equivalent to $P(A|B \cap C) = P(A | C)$

Contrast to Plain Independence. Two events A and B are **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

Example – Tossing Coins

Suppose that Coin 1 has probability of heads 0.3
and Coin 2 has probability of head 0.9.

We choose one of the coins randomly with equal probability and flip that coin 2 times independently. **What is the probability we get all heads?**

C_i = coin i was selected

$$P(HH) = P(HH | C_1) \cdot P(C_1) + P(HH | C_2) \cdot P(C_2) \quad \text{Law of Total Probability (LTP)}$$

$$= P(H|C_1)^2 P(C_1) + P(H | C_2)^2 P(C_2) \quad \text{Conditional Independence}$$

$$= 0.3^2 \cdot 0.5 + 0.9^2 \cdot 0.5 = 0.45$$

Example – Tossing coins (2)

Suppose that Coin 1 has probability of heads 0.3
and Coin 2 has probability of head 0.9.

We choose one coin randomly with equal probability and flip that coin 2 times independently. How does the probability we get all heads compare to $P(H)^2$?

$$P(HH | C_1) = P(H | C_1)^2$$

$$P(HH) \neq P(H)^2$$

$$P(HH) = P(H | C_1)^2 P(C_1) + P(H | C_2)^2 P(C_2) = 0.3^2 \cdot 0.5 + 0.9^2 \cdot 0.5 = 0.45$$

$$P(H) = P(H | C_1) \cdot P(C_1) + P(H | C_2) \cdot P(C_2) = 0.6 \qquad P(HH) = 0.36$$