More on Independence

CSE 312 Spring 26

Lecture 7

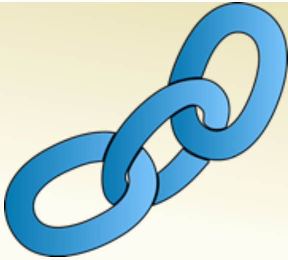
Review - Conditional Probability

Definition. The **conditional probability** of event E **given** an event F happened (assuming $P(F) \neq 0$) is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

If the probability space is uniform, then $P(E|F) = \frac{|E \cap F|}{|F|}$

Review - Chain Rule (general case)



$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} \quad \Longrightarrow \quad \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{B})$$

Theorem. (Chain Rule) For events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$,

$$\mathbb{P}(\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_n) = \mathbb{P}(\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_2|\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_3|\mathcal{A}_1 \cap \mathcal{A}_2)$$

$$\cdots \mathbb{P}(\mathcal{A}_n|\mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_{n-1})$$

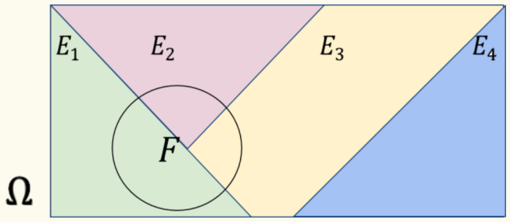
An easy way to remember: We have n tasks and we can do them sequentially, conditioning on the outcome of previous tasks

Review: Conditional & Total Probabilities

- **Conditional Probability**

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad \text{if } P(A) \neq 0$$

- **Law of Total Probability**



$E_1, \ldots, E_n$ partition Ω

$$P(F) = \sum_{i=1}^{n} P(F \cap E_i) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

- **Bayes Theorem**

P(Z|T)

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \qquad \text{if } P(A) \neq 0, P(B) \neq 0$$

Review: Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let $E_1, E_2, \ldots, E_n$ be a partition of the sample space, and F and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

Simple Partition: In particular, if E is an event with non-zero probability, then

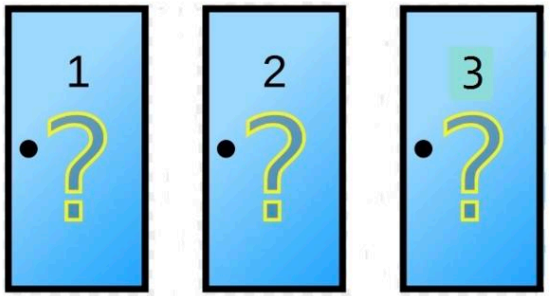
$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

(Z|T)

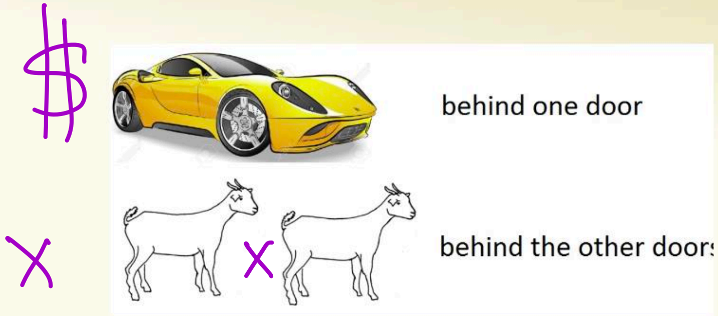
Agenda

- Recap
- The Monty Hall Problem
- Conditional Probability Defines Probability Space
- More on independence
  - Example using Law of Total Probability
  - Independence of many events
  - Independence not always obvious
  - Defining a probability space using independence
  - Independence as an assumption not always justified

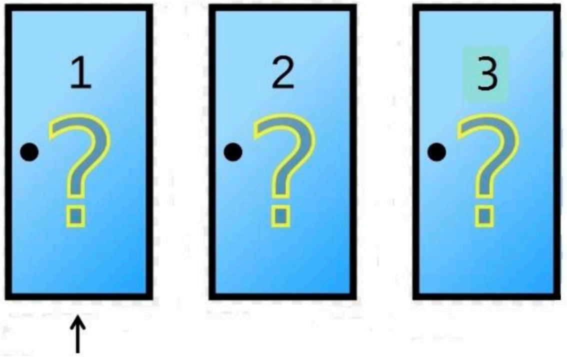
The Monty Hall Problem



Suppose you're on a game show
You're given the choice of three doors

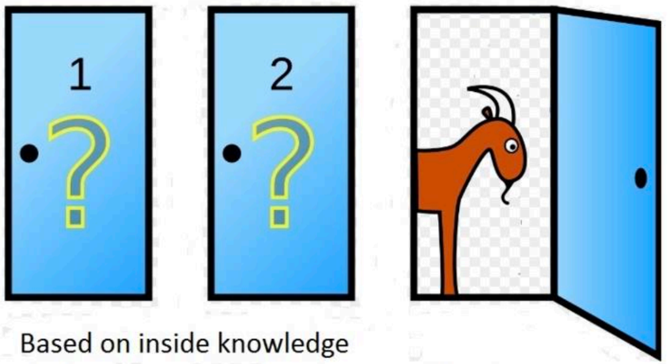


Car equally likely to be behind any of 3 doors



You pick a door, say No. 1

Say, pick uniformly at random



Based on inside knowledge
the host opens another door, say No. 3

If host has two options, picks uniformly at random.

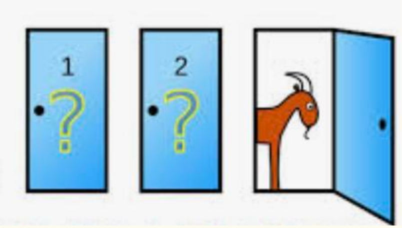
Finally, you are offered the option to switch to door 2. Should you?

In words….

Suppose you're on a game show, and you're given the choice of three doors. Behind one of the doors is a car, behind the other, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to switch to door number 2?" Is it to your advantage to switch your choice of doors?

Assumptions

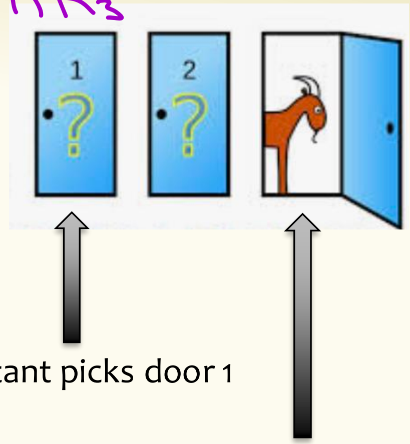
- The player is equally likely to pick each of the three doors, say door 1.
- After the player picks a door, the host **must** open a different door with a goat behind it and offer the player the choice of staying with the original door he selected or switching to the other unopened door
- If the host has two options, he picks one at random.



Should you stay or should you switch?

- P_i = the prize is behind door i.
- C_i= the contestant picks door i,   say door 1.
- R_i = the host reveals door i, say door 3.

$C_1 \cap R_3$



C_1 = Contestant picks door 1

R_3 = Host reveals door 3

$$P(P_2 \cap C_1 \cap R_3) = P(P_2)\,P(C_1 \mid P_2)\,P(R_3 \mid P_2 \cap C_1) = \frac{1}{9}$$

$$P(P_1 \cap C_1 \cap R_3) = P(P_1)\,P(C_1 \mid P_1)\,P(R_3 \mid P_1 \cap C_1) = \frac{1}{18}$$

$$P(C_1 \cap R_3) = P(P_1 \cap C_1 \cap R_3) + P(P_2 \cap C_1 \cap R_3) + P(P_3 \cap C_1 \cap R_3) = \frac{1}{9} + \frac{1}{18} + 0 = \frac{3}{18} = \frac{1}{6}$$

$$\mathbb{P}(\text{prize behind door } 2 \mid C_1 \cap R_3) = \frac{\mathbb{P}(P_2 \cap C_1 \cap R_3)}{\mathbb{P}(C_1 \cap R_3)} = \frac{1/9}{1/6} = \frac{2}{3}$$

$$\mathbb{P}(\text{prize behind door } 1 \mid C_1 \cap R_3) = \frac{\mathbb{P}(P_1 \cap C_1 \cap R_3)}{\mathbb{P}(C_1 \cap R_3)} = \frac{1/18}{1/6} = \frac{1}{3}$$

Should you stay or should you switch (written out)?

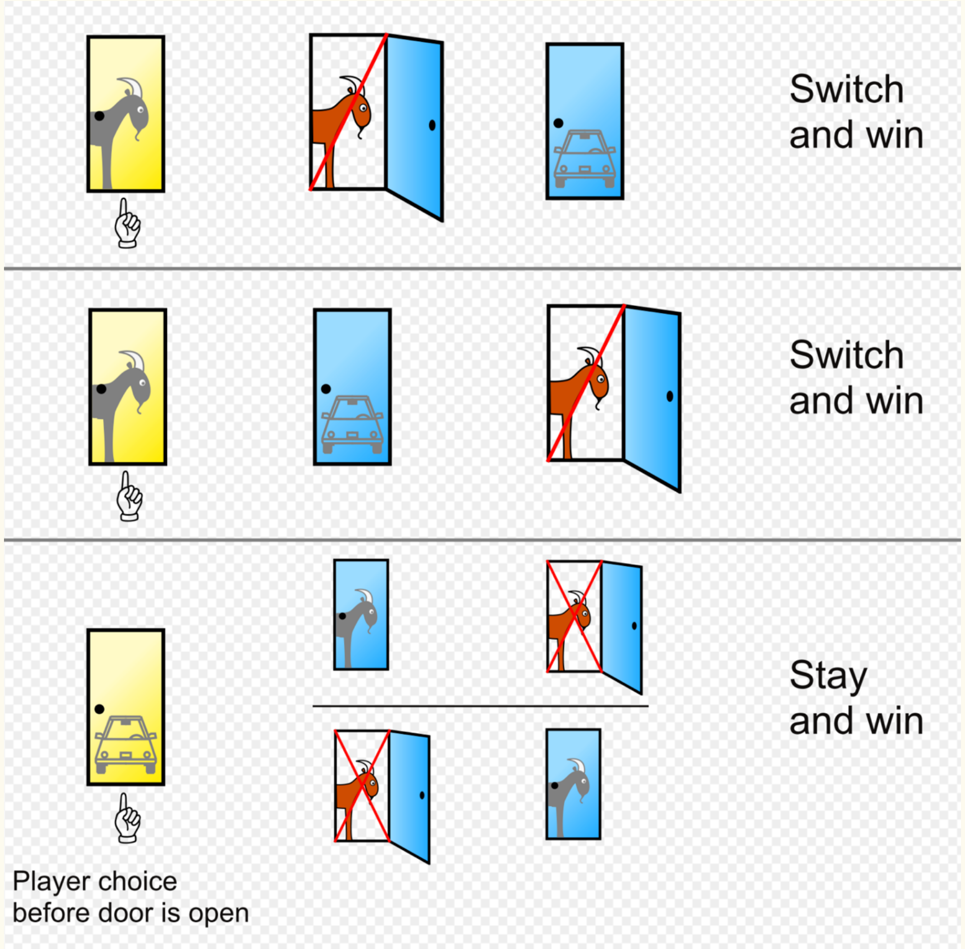
- P_{i} = the prize is behind door i.
- C_{i}= the contestant picks door i, say door 1.
- R_{i} = the host reveals door i, say door 3.
- $\mathbb{P}(P_2 \cap C_1 \cap R_3) = \mathbb{P}(P_2)\mathbb{P}(C_1|P_2)\mathbb{P}(R_3|P_2 \cap C_1) = \frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}$
- $\mathbb{P}(P_1 \cap C_1 \cap R_3) = \mathbb{P}(P_2)\mathbb{P}(C_1|P_1)\mathbb{P}(R_3|P_1 \cap C_1) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18}$
- Since P_1, P_2, P_3 partition the sample space, by the law of total probability
- $\mathbb{P}(C_1 \cap R_3) = \underbrace{\mathbb{P}(P_1 \cap C_1 \cap R_3)}_{\frac{1}{18}} + \underbrace{\mathbb{P}(P_2 \cap C_1 \cap R_3)}_{\frac{1}{9}} + \underbrace{\mathbb{P}(P_3 \cap C_1 \cap R_3)}_{0} = \frac{1}{6}$

$$\mathbb{P}(P_2 \mid C_1 \cap R_3) = \frac{\mathbb{P}(P_2 \cap C_1 \cap R_3)}{\mathbb{P}(C_1 \cap R_3)} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{2}{3}$$

Hence, the contestant would be better off switching to door 3!!

$$\mathbb{P}(P_1 \mid C_1 \cap R_3) = \frac{\mathbb{P}(P_1 \cap C_1 \cap R_3)}{\mathbb{P}(C_1 \cap R_3)} = \frac{\frac{1}{18}}{\frac{1}{6}} = \frac{1}{3}$$

Options for what can happen



Agenda (2)

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Zika Testing – different question

Z = you have Zika
T = you test positive for Zika

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

$P(T \mid Z) = 0.98$
$P(T \mid \bar{Z}) = 0.01$
$P(Z) = 0.005$

What is the probability you test negative (event $\bar{T}$) if you have Zika (event Z)?

$P(\bar{T} \mid Z)$

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$\Pr(\bar{T} \mid Z)$

A) 0.02
B) 0.99
C) 0.02 x 0.005
D) 0.02 x 0.995

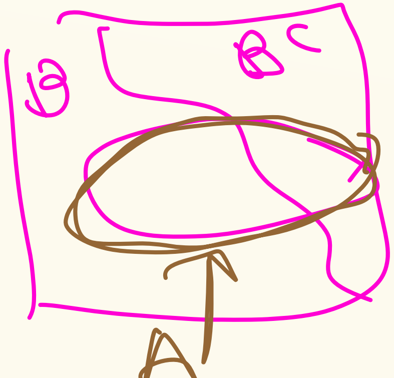
Conditional Probability Define a Probability Space

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

$$P(B|A) + P(B^c|A) = \frac{P(B\cap A)}{P(A)} + \frac{P(B^c\cap A)}{P(A)}$$

$$= \frac{P(B\cap A) + P(B^c\cap A)}{P(A)} = \frac{P(A)}{P(A)}$$

$$= 1$$



Conditional Probability Define a Probability Space (2)

The probability conditioned on A follows the same properties as (unconditional) probability.

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Formally. $(\Omega, \mathbb{P})$ is a probability space + $\mathbb{P}(\mathcal{A}) > 0$

⇒ $(\mathcal{A}, \mathbb{P}(\cdot\,|\mathcal{A}))$ is a probability space

Axiom 1 (Non-negativity): $P(E) \geq 0.$
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If E and F are mutually exclusive,
then $P(E \cup F) = P(E) + P(F)$

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Independence

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

Alternatively,

- If $\mathbb{P}(\mathcal{A}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{B})$
- If $\mathbb{P}(\mathcal{B}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$

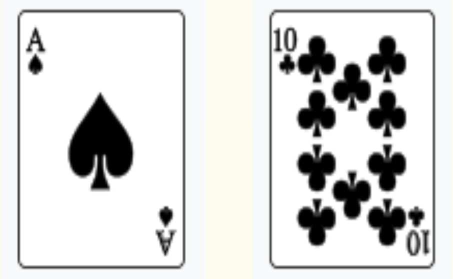
"The probability that $\mathcal{B}$ occurs after observing $\mathcal{A}$" -- Posterior
= "The probability that $\mathcal{B}$ occurs" -- Prior

Are A and B independent?

Have a Standard 52-Card Deck. Shuffle It, and draw the top 2 cards **in order:**

Step 1: Draw a uniformly random card.

Step 2: Draw a uniformly random card from remaining cards.



A: Ace of Spades First

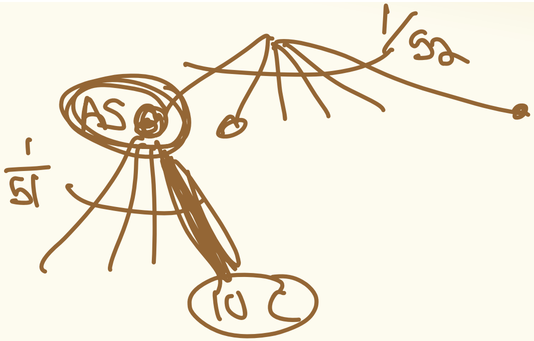
B: 10 of Clubs Second

Are the events A and B independent?

$\mathbb{P}(B|A) = \frac{1}{51}$

$\mathbb{P}(B) = \frac{1}{52}$

Calculation



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$\Pr(2^{nd}\ \text{card} = 10\ \text{clubs} \mid 1^{st}\ card = X)$

A) 1/52
B) 1/51
C) Depends on X
D) (1/52) x (1/51)

2C

$$P(10C\ 2^{nd}) = \sum_{X \text{ first card},\ X \in \{1,\dots,52\}} \Pr(1^{st}\ \text{card}\ X)\, P(2^{nd}\ \text{card}\ 10C \mid 1^{st}\ \text{card}\ X)$$

$$= \begin{cases} \frac{1}{51} & X \neq 10C \\ 0 & X = 10C \end{cases}$$

$$= \sum_{X \neq 10C,\ X \in \{1 \dots 52\}} \frac{1}{52} \cdot \frac{1}{51} = 51 \cdot \frac{1}{52} \cdot \frac{1}{51}$$

$$= \frac{1}{52}$$

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Multiple Events – Mutual Independence

Definition. Events $A_1, \dots, A_n$ are **mutually independent** if for every non-empty subset $I \subseteq \{1, \dots, n\}$, we have

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i).$$

$A_1, A_2, \dots, A_{10}$

$$P(A_1 \cap A_5 \cap A_7) = P(A_1)P(A_5)P(A_7)$$

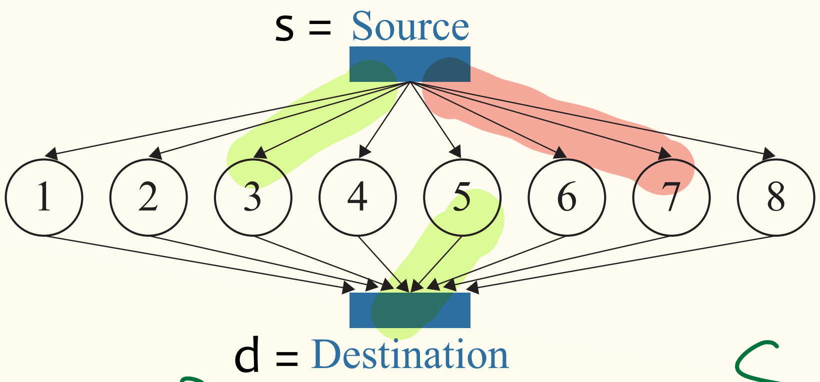
Multiple Events – Mutual Independence - corollary

if $A_1, \ldots, A_n$ are **mutually independent** then for any choice of $\mathrm{B}_i \in \{A_i, {A_i}^{\mathrm{c}}\}, \quad i = 1..\mathrm{n}$ and every non-empty subset $I \subseteq \{1, \ldots, n\}$, we have

$$P\left(\bigcap_{i \in I} \mathrm{B}_i\right) = \prod_{i \in I} P(\mathrm{B}_i).$$

An example

You are routing a packet from the source to the destination.
But each of the 16 edges in the network works with probability p, independently.



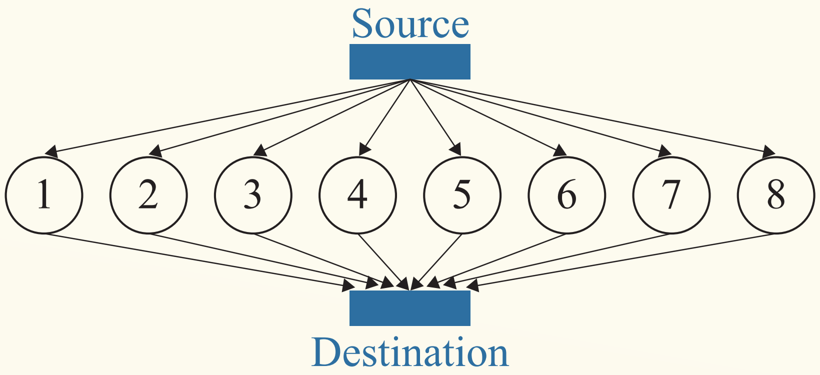
Q: Pr(edge (s,3) works, edge (5,d) works, edge (s,7) doesn't work)?

$= P(A)P(B)P(C) =$

$p \quad p \quad 1-p$

An example (2)

You are routing a packet from the source to the destination.
But each of the 16 edges in the network works with probability p, independently.



Q: What is the probability that you can get the packet from the source to the destination?

Let's break it down

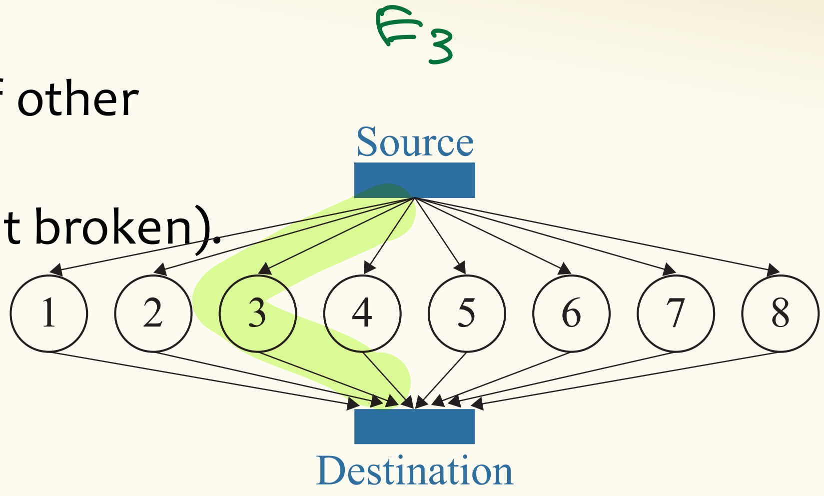
Each edge works with probability p, independently of other edges. There are 8 paths.
Let E_i denote the event that the i^{th} path is usable (not broken).

Q1: What is $\boldsymbol{P}\{E_i\}$?

$P((s,i)$ works and (i,d) works$)$
$= p^2$

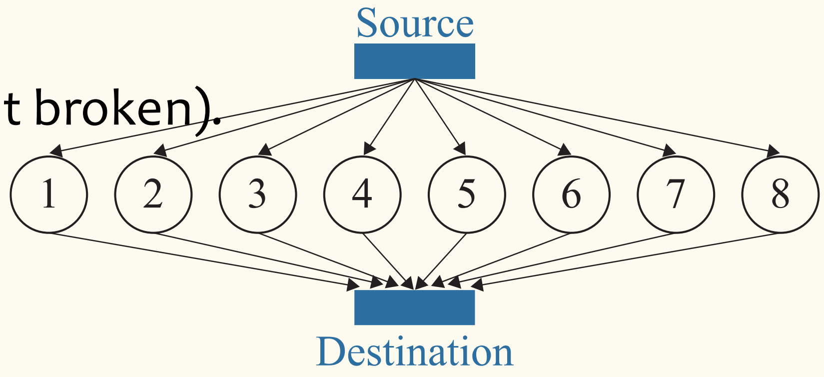
Q2: What is $\boldsymbol{P}\{\overline{E_i}\}$?

$1-p^2$



More calculations

Each edge works with probability p, independently of other edges. There are 8 paths.
Let E_i denote the event that the i^{th} path is usable (not broken).



$\boldsymbol{P}\{E_i\} = p^2$ $\qquad\qquad$ $\boldsymbol{P}\{\overline{E_i}\} = 1 - p^2$

What is $\boldsymbol{P}\{\text{Can get from source to destination}\}$?

$\boldsymbol{P}\{\text{Can get from source to destination}\} =$

Pr(at least one E_i works) = 1 − Pr(no path works)

$= 1 - \Pr(\overline{E_1} \cap \overline{E_2} \cap \overline{E_3} \cdots \cap \overline{E_8})$

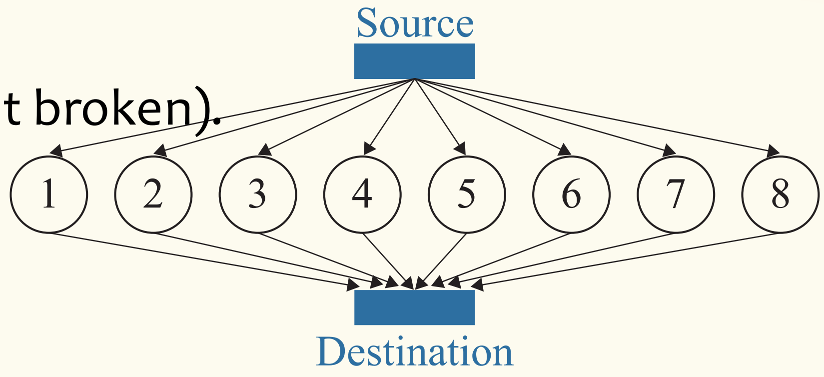
$= 1 - P(\overline{E_1})P(\overline{E_2}) \cdots P(\overline{E_8})$

$(1-p^2)(1-p^2) \cdots (1-p^2)$

$= 1 - (1-p^2)^8$

Answer

Each edge works with probability p, independently of other edges. There are 8 paths.
Let E_i denote the event that the i^{th} path is usable (not broken).



$\boldsymbol{P}\{E_i\} = p^2$ $\qquad\qquad$ $\boldsymbol{P}\{\overline{E_i}\} = 1 - p^2$

What is $\boldsymbol{P}\{\text{Can get from source to destination}\}$?

$$\boldsymbol{P}\{\text{Can get from source to destination}\} = \boldsymbol{P}\{\text{At least one path works}\}$$

$$= \boldsymbol{P}\{E_1 \cup E_2 \cup \cdots \cup E_8\}$$

$$= 1 - \boldsymbol{P}\{\text{All paths are broken}\}$$

$$= 1 - \boldsymbol{P}\{\overline{E_1}\} \cdot \boldsymbol{P}\{\overline{E_2}\} \cdots \boldsymbol{P}\{\overline{E_8}\} = 1 - (1 - p^2)^8$$

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Defining a probability space

Often probability space $(\Omega, \mathbb{P})$ is **defined** using independence

Example – Biased coin

We have a biased coin comes up Heads with probability 2/3; Each flip is independent of all other fips. Suppose it is tossed 3 times.

$$\mathbb{P}(HHH) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \left(\frac{2}{3}\right)^3$$

$$\mathbb{P}(TTT) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \left(\frac{1}{3}\right)^3$$

$$\mathbb{P}(HTT) = \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}$$

Example – Biased coin – let's do a calculation

We have a biased coin comes up Heads with probability 2/3, independently of other flips. Suppose it is tossed 3 times.

$\mathbb{P}(2 \text{ heads in 3 tosses}) =$

$P(\{HHT, HTH, THH\})$

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A) $(2/3)^2\ 1/3$
B) $2/3$
C) $3\ (2/3)^2\ 1/3$
D) $(1/3)^2$