Law of Total Probability, Bayes Rule and More on Independence

CSE 312 Spring 26

Lecture 6

Thank you for your feedback!!!

Some people mentioned that I was going too fast in parts and would like more examples in class.

How do we address this?

What I can do and what I will try to do:

- *I will try to be more consistent in my pacing and go over the examples a little bit slower.*
- *I will try to give more examples.*

What you can do:

- *Slow me down! Ask questions!!!*
- *Give me feedback between classes. (I've added feedback as a category on Ed.)*
- *Do the concept checks!!*
- *Please come to office hours.*
- *Ask the 312 Learning Assistant to explain anything that is confusing.*
- *Form study groups and go over all the section problems together.*
- *Do the readings.*

Agenda

- Recap
- Law of Total Probability
- Bayes Theorem
- More on Independence

Review - Probability

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips:
  $E = \{HH, HT, TH\}$
- Rolling an even number on a die :
  $E = \{2, 4, 6\}$

Review - Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events.

Axiom 1 (Non-negativity): $P(E) \geq 0$
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If E and F are mutually exclusive events,
then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Review: Equally Likely Outcomes

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$, then

$$P(E) = \frac{|E|}{|\Omega|}$$

Examples:

- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$

$$P(E) = \frac{3}{4}$$

- Rolling an even number on a die : $E = \{2, 4, 6\}$

$$P(E) = \frac{3}{6} = \frac{1}{2}$$

Review - Conditional Probability

Definition. The **conditional probability** of event E **given** an event F happened (assuming $P(F) \neq 0$) is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

A relevant example

When you ask ChatGPT or Gemini or any of them a question, they are not “looking up” the answer. They are calculating a conditional probability distribution over a vocabulary of ``tokens”. It is answering the question:

$$Pr(w_n | w_1, w_2, \dots, w_{n-1})\ ?$$

dog 0.9
cat 0.1

``Given the words that have come before, what is the probability of the next word?”

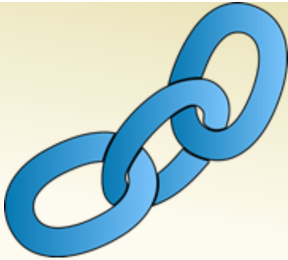
Review - Conditional Probability (uniform case)

Definition. The **conditional probability** of event E **given** an event F happened (assuming $P(F) \neq 0$) is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

If the probability space is uniform, then $P(E|F) = \frac{|E \cap F|}{|F|}$

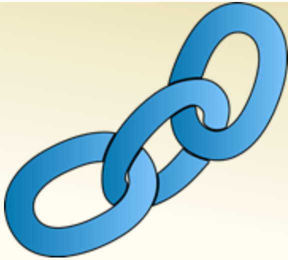
Review - Chain Rule



$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} \quad \Rightarrow \quad \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{B})$$

$$P(B)\,P(A|B)$$

Review - Chain Rule (general case)



$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} \quad \Rightarrow \quad \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{B})$$

Theorem. (Chain Rule) For events $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$,

$$\mathbb{P}(\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_n) = \mathbb{P}(\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_2|\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_3|\mathcal{A}_1 \cap \mathcal{A}_2)$$

$$\cdots \mathbb{P}(\mathcal{A}_n|\mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_{n-1})$$

An easy way to remember: We have n tasks and we can do them sequentially, conditioning on the outcome of previous tasks

Review - Independence

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

Alternatively,

- If $\mathbb{P}(\mathcal{A}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{B})$
- If $\mathbb{P}(\mathcal{B}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$

if A & B are indep.

then so are A and $\overline{B}$

$\overline{A}$ B

$\overline{A}$ $\overline{B}$

Agenda (2)

- Recap
- Law of Total Probability
- Bayes Theorem
- More on independence

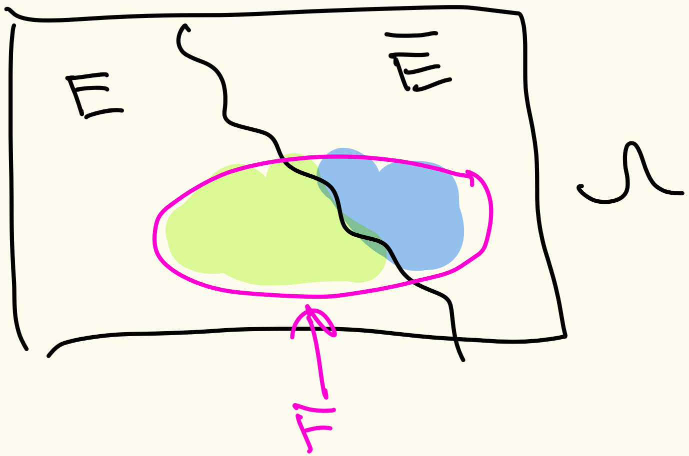
Law of Total Probability

For any sets E and F:

$$F = (F \cap E) \cup (F \cap \overline{E})$$

$$\boldsymbol{P}\{F\} = \boldsymbol{P}\{F \cap E\} + \boldsymbol{P}\{F \cap \overline{E}\}$$

$$= \boldsymbol{P}\{F|E\} \cdot \boldsymbol{P}\{E\} + \boldsymbol{P}\{F|\overline{E}\} \cdot \boldsymbol{P}\{\overline{E}\}$$



Law of Total Probability (General case)

For any sets E and F:

$$F = (F \cap E) \cup (F \cap \overline{E})$$

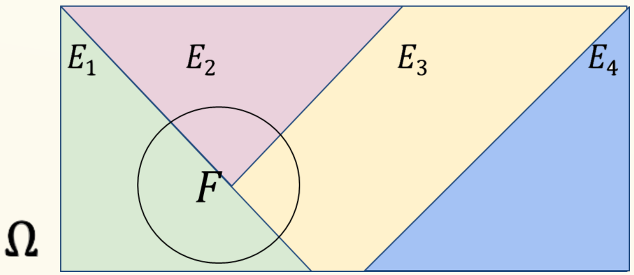
$$\boldsymbol{P}\{F\} = \boldsymbol{P}\{F \cap E\} + \boldsymbol{P}\{F \cap \overline{E}\}$$

$$= \boldsymbol{P}\{F|E\} \cdot \boldsymbol{P}\{E\} + \boldsymbol{P}\{F|\overline{E}\} \cdot \boldsymbol{P}\{\overline{E}\}$$

Generalizing, we have:

Theorem: [Law of Total Probability]
Let $E_1, E_2, \dots, E_n$ partition the state space Ω. Then:

$$\boldsymbol{P}\{F\} = \sum_{i=1}^{n} \boldsymbol{P}\{F \cap E_i\} = \sum_{i=1}^{n} \boldsymbol{P}\{F|E_i\} \cdot \boldsymbol{P}\{E_i\}$$

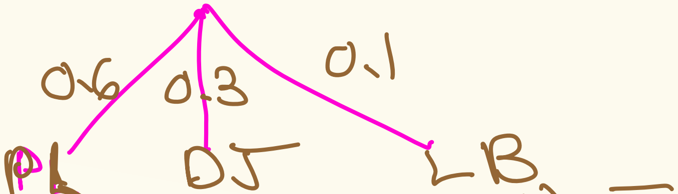


Dance party?

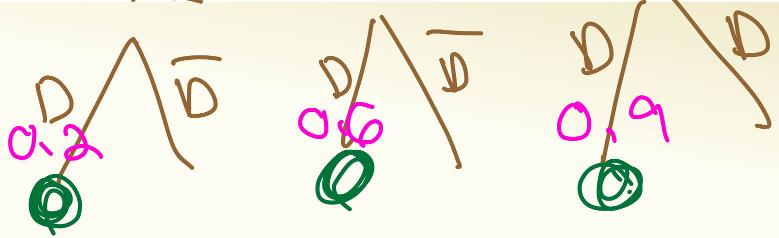
- Your friend is having a party. The music will either be:
  - Playlist (PL) – probability 0.6
  - DJ - probability 0.3
  - Live band (LB) – probability 0.1

- You are very picky about what music you will dance to. If they go with their playlist, you will dance with probability 0.2, if they go with a DJ, you'll dance with probability 0.6 and if they go with a live band, you'll dance with probability 0.9.

- What's the probability you dance and the music is a DJ?



Dance Party (2)



$Pr(PL) = 0.6$
$Pr(DJ) = 0.3$
$Pr(LB) = 0.1$

D=Dance
$Pr(D \mid PL) = 0.2$
$Pr(D \mid DJ) = 0.6$
$Pr(D \mid LB) = 0.9$

https://pollev.com/ annakarlin185

What is Pr (Dance and DJ)?

$= P(DJ)P(D \mid DJ)$

A) 0.3
B) 0.3 x 0.6
C) 0.3^2
D) 0.6

What is the probability you dance?

$P(D) = P(PL)P(D|PL) + P(DJ)P(D|DJ)$
$+ P(LB)P(D|LB)$

<u>Theorem</u>: **[Law of Total Probability]**

Let $E_1, E_2, \ldots, E_n$ partition the state space Ω. Then:

$$\boldsymbol{P}\{F\} = \sum_{i=1}^{n} \boldsymbol{P}\{F \cap E_i\} = \sum_{i=1}^{n} \boldsymbol{P}\{F|E_i\} \cdot \boldsymbol{P}\{E_i\}$$

Dance party (4)

- Your friend is having a party. The music will either be:
  - Playlist (PL) – probability 0.6
  - DJ - probability 0.3
  - Live band (LB) – probability 0.1

- You are very picky about what music you will dance to. If they go with their playlist, you will dance with probability 0.2, if they go with a DJ, you'll dance with probability 0.6 and if they go with a live band, you'll dance with probability 0.9.

- What's the probability you dance?

$$\Pr(\text{Dance}) = \Pr(D \mid PL)\Pr(PL) + \Pr(D \mid DJ)\Pr(DJ) + \Pr(D \mid LB)\Pr(LB)$$

Agenda (3)

- Recap
- Law of Total Probability
- Bayes Theorem
- More on Independence

Our First Machine Learning Task: Spam Filtering

`Subject: “FREE $$$ CLICK HERE”`

Suppose you know that 80% of emails you receive are spam.

So a priori, our belief is that any email has an 80% chance of being spam.

How do you update that belief when you see that the subject line contains the phrase “FREE $$$”?

Bayes Theorem

A formula to let us "reverse" the conditional.

Theorem. (Bayes Rule) For events A and B, where $P(A), P(B) > 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

chain rule. $P(A \cap B)$

$P(A)$ is called the **prior** (our belief without knowing anything)

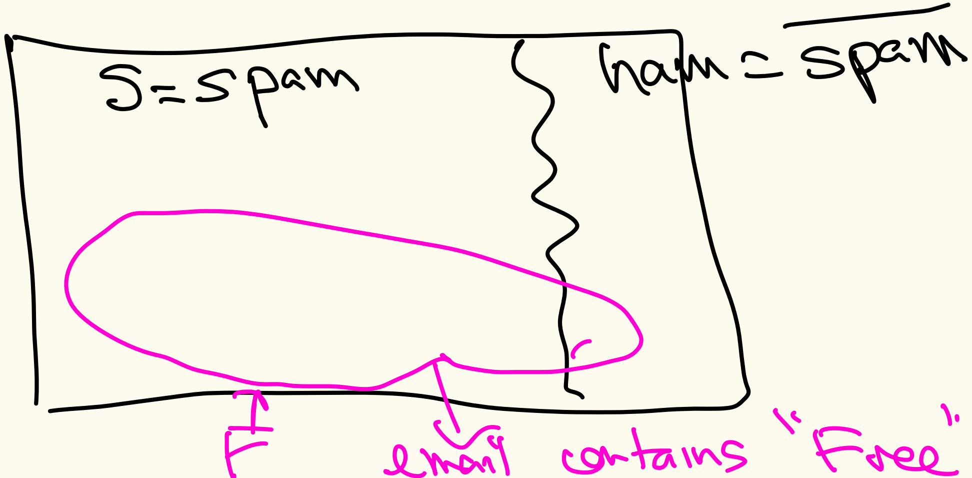
$P(A|B)$ is called the **posterior** (our belief after learning B)

Back to Spam Filtering - 1

Subject: “FREE $$$ CLICK HERE”

What is the probability this email is spam, given the subject contains “FREE”?
Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word “FREE” in the subject.
- 70% of spam emails contain the word “FREE” in the subject.
- 80% of emails you receive are spam.



Back to Spam Filtering - 2

Subject: “FREE $$$ CLICK HERE”

What is the probability this email is spam, given the subject contains “FREE”?
Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word “FREE” in the subject. $P(F|\bar{S}) = 0.1$
- 70% of spam emails contain the word “FREE” in the subject. $P(F|S) = 0.7$
- 80% of emails you receive are spam. $P(S) = 0.8$

You receive a random email:

- Let S be event that the email is spam
- Let F be the event that the email contains the word “FREE”.

Back to Spam Filtering - 3

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Subject: "FREE $$$ CLICK HERE"
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What is the probability this email is spam, given the subject contains "FREE"?

Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word "FREE" in the subject.
- 70% of spam emails contain the word "FREE" in the subject.
- 80% of emails you receive are spam.

$$P(F|\overline{S}) = 0.1$$
$$P(F|S) = 0.7$$
$$P(S) = 0.8$$

You receive a random email:

- Let S be event that the email is spam
- Let F be the event that the email contains the word "FREE".

$$P(S|F) = \frac{P(F|S)\,P(S)}{P(F)}$$

Back to Spam Filtering - 4

Subject: "FREE $$$ CLICK HERE"

What is the probability this email is spam, given the subject contains "FREE"?
Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word "FREE" in the subject. $P(F|\overline{S}) = 0.1$
- 70% of spam emails contain the word "FREE" in the subject. $P(F|S) = 0.7$
- 80% of emails you receive are spam. $P(S) = 0.8$

$P(\overline{S}) = 0.2$

You receive a random email:

- Let S be event that the email is spam
- Let F be the event that the email contains the word "FREE".

$P(\overline{S}) = 1 - P(S)$

By Bayes Rule, $P(S|F) = \dfrac{P(F|S)P(S)}{P(F)} = \dfrac{0.7 \cdot 0.8}{P(F)}$

$P(S|F) = 0.9655$

$P(F) = P(S)P(F|S) + P(\overline{S})P(F|\overline{S})$

$= 0.8 \cdot 0.7 + 0.2 \cdot 0.1$

Back to Spam Filtering - 5

Subject: "FREE $$$ CLICK HERE"

What is the probability this email is spam, given the subject contains "FREE"?

Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word "FREE" in the subject.
- 70% of spam emails contain the word "FREE" in the subject.
- 80% of emails you receive are spam.

$$P(F|\overline{S}) = 0.1$$
$$P(F|S) = 0.7$$
$$P(S) = 0.8$$

$$P(F|S) = \frac{P(F \cap S)}{P(S)}$$

You receive a random email:

- Let S be event that the email is spam
- Let F be the event that the email contains the word "FREE".

By Bayes Rule, $P(S|F) = \dfrac{P(F|S)P(S)}{P(F)}$

By the Law of Total Probability

$$P(F) = \underbrace{P(F|S)P(S)}_{P(F \cap S)} + \underbrace{P(F|\overline{S})P(\overline{S})}_{P(F \cap \overline{S})}$$

Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let $E_1, E_2, \ldots, E_n$ be a partition of the sample space, and F and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

Simple Partition: In particular, if E is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

Example – Zika Testing

Usually no or mild symptoms (rash); sometimes severe symptoms (paralysis).

During pregnancy: may cause birth defects.

Suppose you took a Zika test, and it returns "positive", what is the likelihood that you actually have the disease?

- Tests for diseases are rarely 100% accurate.

This example and following slides are from Lisa Yan (Stanford).

Example – Zika Testing (1)

Z **have Zika**
T **test positive**

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test yields a "false positive" 1% of the time
- 0.5% of the US population has Zika.

$P(T|Z) = 0.98$

$P(T|\bar{Z}) = 0.01$

$P(Z) = 0.005$

$P(\bar{Z}) = 0.995$

What is the probability a random person in the US has Zika (event *Z*) given that they test positive (event *T*)?

$$P(Z|T) = \frac{P(T|Z)P(Z)}{P(T)}$$

$$P(T) = P(T \cap Z) + P(T \cap \bar{Z})$$
$$= P(T|Z)P(Z) + P(T|\bar{Z})P(\bar{Z})$$

Example – Zika Testing (2)

$\boldsymbol{Z}$ **have Zika**
$\boldsymbol{T}$ **test positive**

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $\boldsymbol{P(T|Z) = 0.98}$
- However, the test may yield a "false positive" 1% of the time $\boldsymbol{P(T|Z^c)}$**= 0.01**
- 0.5% of the US population has Zika. $\boldsymbol{P(Z)}$**=0.005**

What is the probability you have Zika (event Z) given that you test positive (event T)?

By Bayes Rule, $\boldsymbol{P(Z|T) = \dfrac{P(T|Z)P(Z)}{P(T)}}$

Example – Zika Testing (3)

Z have Zika
T test positive

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $P(T|Z) = 0.98$
- However, the test may yield a "false positive" 1% of the time $P(T|Z^c)$=0.01
- 0.5% of the US population has Zika. $P(Z)$=0.005

What is the probability you have Zika (event Z) given that you test positive (event T)?

By Bayes Rule, $P(Z|T) = \dfrac{P(T|Z)P(Z)}{P(T)} = \dfrac{0.98 \cdot 0.005}{P(T)} \approx 0.3297$

By the Law of Total Probability, $P(T) = P(T|Z)P(Z) + P(T|Z^c)P(Z^c)$

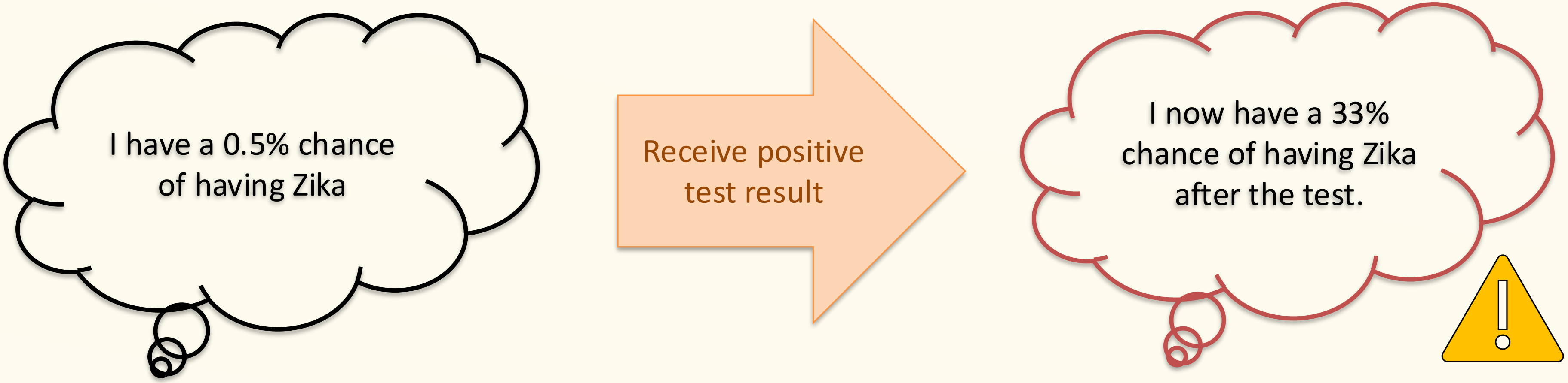
$$= 0.98 \cdot 0.005 + 0.01 \cdot 0.995$$

Philosophy – Updating Beliefs

While it's not 98% that you have the disease, your beliefs changed significantly

Z = you have Zika

T = you test positive for Zika



Prior: $P(Z)$

Posterior: $P(Z|T)$

Example – Zika Testing (5)

Have zika blue, don't pink

Picture below gives us the following Zika stats

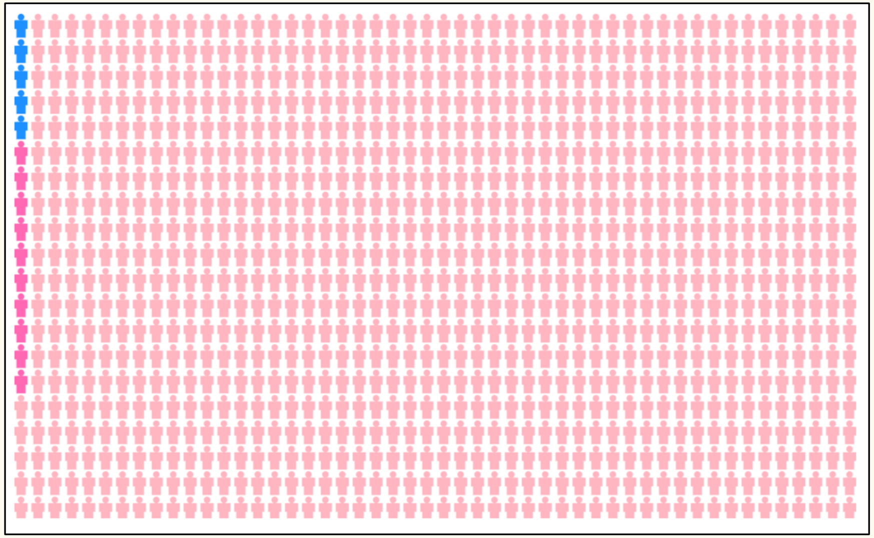
- A test is 100% effective at detecting Zika ("true positive").
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika. 5% have it.

$P(T|Z) = 5/5 = 1$

$P(T|Z^c) = 10/995 \approx 0.01$

$P(Z) = \frac{5}{1000} = 0.005$

What is the probability you have Zika (event Z) given that you test positive (event T)?



Suppose we had 1000 people:

- 5 **have Zika** and and **test positive**
- 985 **do not have Zika** and **test negative**
- 10 **do not have Zika** and **test positive**

Example – Zika Testing (5)

Have zika blue, don't pink

Picture below gives us the following Zika stats

- A test is 100% effective at detecting Zika ("true positive").
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika. 5% have it.

$$P(T|Z) = 5/5 = 1$$

$$P(T|Z^c) = 10/995$$

$$P(Z) = \frac{995}{1000} = 0.005$$

What is the probability you have Zika (event Z) given that you test positive (event T)?

Suppose we had 1000 people:

- 5 **have Zika** and and **test positive**
- 985 **do not have Zika** and **test negative**
- 10 **do not have Zika** and **test positive**

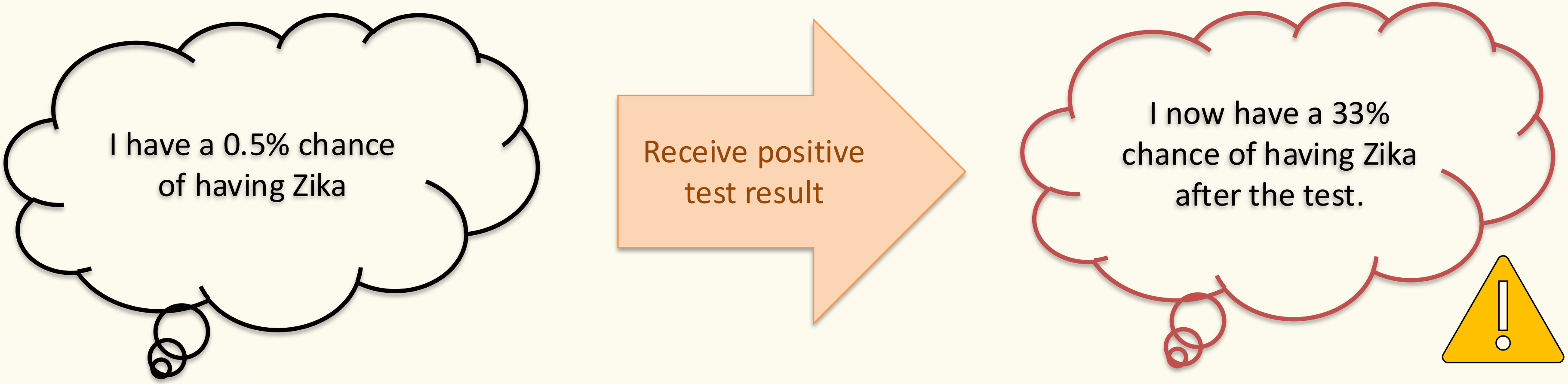
$$\frac{5}{5+10} = \frac{1}{3} \approx 0.33$$

Philosophy – Updating Beliefs - should be clearer

While it's not 98% that you have the disease, your beliefs changed **drastically**

Z = you have Zika

T = you test positive for Zika



Prior: P(Z)

Posterior: P(Z|T)