

# Am I going too fast in lecture? Your responses are anonymous.

<https://pollev.com/annakarlin185>

- A. Yes, you are going **way too fast**.
- B. Yes, you are going a **little bit too fast**.
- C. No, your pace is **about right**.
- D. No, your pace is **too slow**.

3%

54%

35%

11%

# The Birthday Paradox, Conditional Probability and more

CSE 312 Spring 26  
Lecture 5

Many of the slides in this lecture were created by Mor Harchol-Balter for her book: "Introduction to Probability for Computing", Harchol-Balter '24

# Review - Probability

**Definition.** A **sample space**  $\Omega$  is the set of all possible outcomes of an experiment.

**Definition.** An **event**  $E \subseteq \Omega$  is a subset of possible outcomes.

Examples:

- Single coin flip:  $\Omega = \{H, T\}$
- Two coin flips:  $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die:  $\Omega = \{1, 2, 3, 4, 5, 6\}$

Examples:

- Getting at least one head in two coin flips:  
 $E = \{HH, HT, TH\}$
- Rolling an even number on a die :  
 $E = \{2, 4, 6\}$

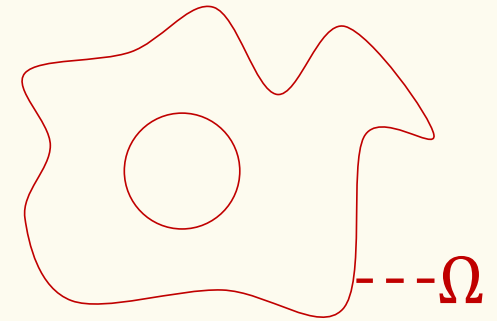
# Probability space - more

Either finite or infinite countable (e.g., integers)

**Definition.** A (discrete) **probability space** is a pair  $(\Omega, \mathbb{P})$  where:

- $\Omega$  is a set called the **sample space**.
- $\mathbb{P}$  is the **probability measure**, a function  $\mathbb{P}: \Omega \rightarrow [0,1]$  such that:
  - $\mathbb{P}(\omega) \geq 0$  for all  $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Set of possible elementary outcomes



Specify Likelihood (or probability) of each elementary outcome

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

# Review Axioms of Probability

Let  $\Omega$  denote the sample space and  $E, F \subseteq \Omega$  be events.

**Axiom 1 (Non-negativity):**  $P(E) \geq 0$

**Axiom 2 (Normalization):**  $P(\Omega) = 1$

**Axiom 3 (Countable Additivity):** If  $E$  and  $F$  are mutually exclusive events, then  $P(E \cup F) = P(E) + P(F)$

**Corollary 1 (Complementation):**  $P(E^c) = 1 - P(E)$

**Corollary 2 (Monotonicity):** If  $E \subseteq F$ ,  $P(E) \leq P(F)$

**Corollary 3 (Inclusion-Exclusion):**  $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

# Review: Equally Likely Outcomes

If  $(\Omega, P)$  is a **uniform** probability space, then for any event  $E \subseteq \Omega$ , then

$$P(E) = \frac{|E|}{|\Omega|}$$

Examples:

- Two coin flips:  $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die:  $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events:

- Getting at least one head in two coin flips:  $E = \{HH, HT, TH\}$

$$P(E) = \frac{3}{4}$$

- Rolling an even number on a die :  $E = \{2, 4, 6\}$

$$P(E) = \frac{3}{6} = \frac{1}{2}$$

## Example: Birthday “Paradox”

(1, 3, 33)  
↑ ↑ ↑

Suppose we have a collection of  $n$  people in a room. What is the probability that at least 2 people share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

Say the people’s names are 1, 2, 3,..n

**Sample space:** all possible assignments of birthdays to these people =

$$\Omega = \{(b_1, b_2, \dots, b_n) : b_i \in \{1, \dots, 365\}\}$$

( $b_i = 25$  means person  $i$  was born on the 25<sup>th</sup> day of the year)

$$|\Omega| = 365 \cdot 365 \cdots 365 = 365^n$$

**Uniform probability space:**  $\mathbb{P}(\omega) = \frac{1}{365^n}$

## Example: Birthday "Paradox"

$$P(\bar{E}) = \frac{|\bar{E}|}{|\Omega|}$$

$E = \{ \text{at least 2 people share a birthday} \}$

$\bar{E} = \{ \text{no pair of people share a birthday} \}$

What is  $\mathbb{P}(E)$ ?

$$P(E) = 1 - P(\bar{E})$$

$$= 1 - \frac{365 \cdot 364 \cdots 365 - (n-1)}{365^n} = 1 - \frac{365!}{(365-n)! (365)^n}$$

$$n = 23$$

$$P(E) > 0.5$$

$$n = 60$$

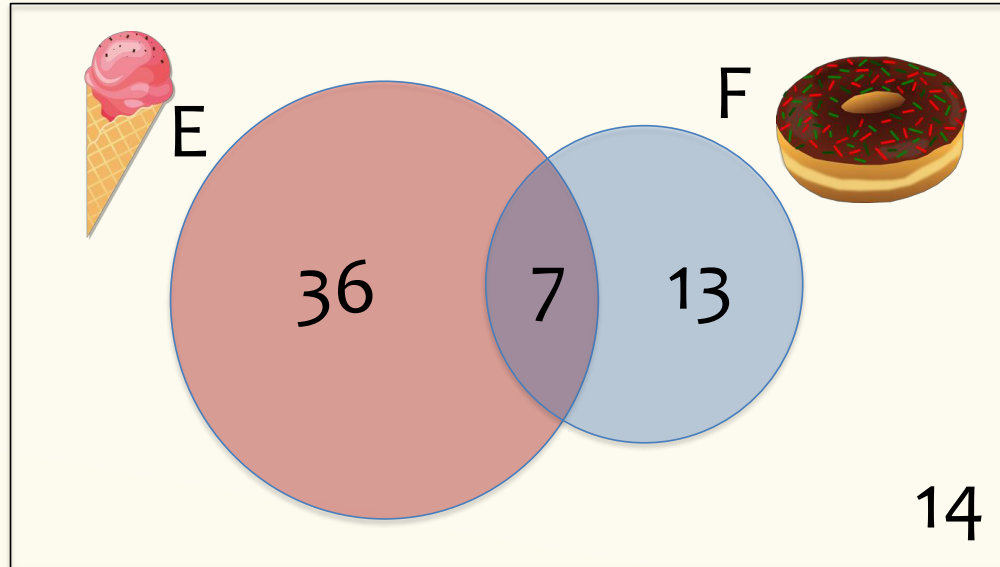
$$\Pr(E) > 0.99$$

# Agenda (1)

- Conditional Probability
- Chain Rule
- Independence
- Law of Total Probability
- Bayes Theorem

# Conditional Probability (Idea)

$$36 + 7 + 13 + 14 = 70$$



What's the probability that a uniformly random person likes ice cream **given** they like donuts?

$$\frac{7}{13+7} = \frac{7}{20}$$

# Conditional Probability

**Definition.** The **conditional probability** of event  $E$  **given** an event  $F$  happened (assuming  $P(F) > 0$ ) is

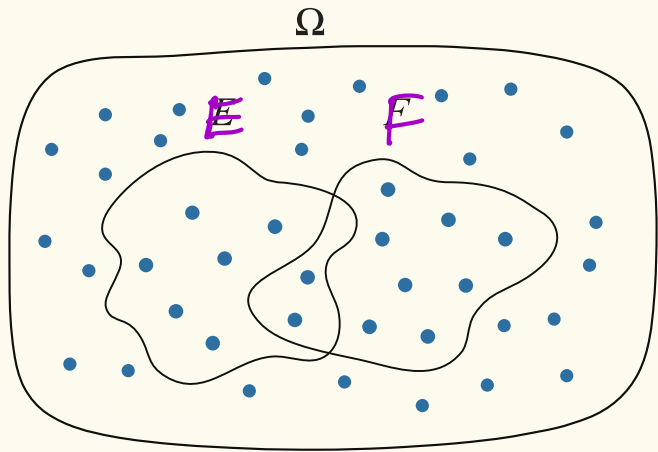
$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

*given that  
conditioned on*

# Conditional Probability

**Definition.** The **conditional probability** of event  $E$  given an event  $F$  happened (assuming  $P(F) \neq 0$ ) is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$



42 dots

Two equivalent views for **equally likely outcomes**:

$$P\{E | F\} = \frac{2}{10} \quad (\text{of the 10 outcomes in set } F, \text{ only 2 of these are in set } E)$$

$$P\{E | F\} = \frac{P\{E \cap F\}}{P\{F\}} = \frac{\cancel{\frac{2}{42}}}{\frac{10}{42}} = \frac{2}{10}$$

# Conditional Probability on Events



Q: What is  $P\{\text{both are colts}\}$ ?

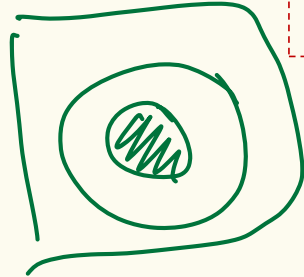
The offspring of a horse is called a foal.  
Horse couples have one foal at a time and  
each foal is either a “colt” or a “filly.”

$$\frac{1}{4}$$

We're told that a horse couple had 2 foals  
and that each outcome is equally likely. ~~We~~  
~~are also told that at least one of these is a~~  
~~colt~~

$$\Omega = \{CC, CF, FC, FF\}$$

# Conditional Probability on Events



**Definition.** The **conditional probability** of event  $E$  **given** an event  $F$  happened (assuming  $P(F) \neq 0$ ) is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

The offspring of a horse is called a foal. Horse couples have one foal at a time and each foal is either a “colt” or a “filly.”

We’re told that a horse couple had 2 foals and that each outcome is equally likely. We are also told that at least one of these is a colt.

**Q:** What is  $P\{\text{both are colts} \mid \geq 1 \text{ colt}\}$ ?

$$\frac{P(\text{both are colts and } \geq 1 \text{ colt})}{P(\geq 1 \text{ colt})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

# Conditional Probability on Events - Answer



The offspring of a horse is called a foal.  
Horse couples have one foal at a time and each foal is either a “colt” or a “filly.”

We’re told that a horse couple had 2 foals and that each outcome is equally likely. We are also told that at least one of these is a colt.

$$\begin{aligned} & P\{\text{both are colts} \mid \geq 1 \text{ colt}\} \\ &= \frac{P\{\text{both are colts} \& \geq 1 \text{ colt}\}}{P\{\geq 1 \text{ colt}\}} \\ &= \frac{P\{\text{both are colts}\}}{P\{\geq 1 \text{ colt}\}} \\ &= \frac{1}{\frac{3}{4}} \\ &= \frac{1}{3} \end{aligned}$$

# Conditional Probability on Events - picture



$$P\{\text{both are colts} \mid \geq 1 \text{ colt}\} \\ = \frac{1}{3}$$

The offspring of a horse is called a foal.  
Horse couples have one foal at a time and each foal is either a “colt” or a “filly.”

We’re told that a horse couple had 2 foals and that each outcome is equally likely. We are also told that at least one of these is a colt.

	Colt	Filly
Colt	✓	
Filly		

## Conditional Probability in a uniform probability space

**Definition.** The **conditional probability** of event  $E$  **given** an event  $F$  happened (assuming  $P(F) \neq 0$ ) is

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{|E \cap F|}{|\Omega|}}{\frac{|F|}{|\Omega|}}$$

If the probability space is uniform, then  $P(E|F) = \frac{|E \cap F|}{|F|}$

## Agenda (2)

- Conditional Probability
- Chain Rule
- Independence
- Law of Total Probability
- Bayes Theorem

# Chain Rule

If  $P\{E_1 \cap E_2\} > 0$ , then:

$$P\{E_2|E_1\} = \frac{P\{E_1 \cap E_2\}}{P\{E_1\}}$$

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Equivalently, we can write:

If  $P\{E_1 \cap E_2\} > 0$ , then:

$$P\{E_1 \cap E_2\} = P\{E_1\} \cdot P\{E_2|E_1\}$$

$$\text{Likewise: } P\{E_1 \cap E_2\} = P\{E_2\} \cdot P\{E_1|E_2\}$$

Probability  
outcome  
is in both  
 $E_1$  and  $E_2$

Probability  
outcome  
is in  $E_1$

Probability  
outcome  
is in  $E_2$  given  
that it's in  $E_1$

# Chain Rule for Conditioning – general case.

If  $P\{E_1 \cap E_2\} > 0$ , then:

$$P\{E_1 \cap E_2\} = P\{E_1\} \cdot P\{E_2|E_1\}$$

This can be generalized!

Theorem: [Chain Rule for Conditioning]

If  $P\{E_1 \cap E_2 \cap E_3 \cap \dots \cap E_n\} > 0$ , then

$$P\{E_1 \cap E_2 \cap E_3 \cap \dots \cap E_n\}$$

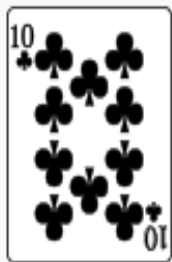
$$= P\{E_1\} \cdot P\{E_2 | E_1\} \cdot P\{E_3 | E_1 \cap E_2\} \cdots P\{E_n | E_1 \cap E_2 \cap E_3 \cap \dots \cap E_{n-1}\}$$

## Chain Rule Example

$\Omega = \{\text{all } 52! \text{ perms of cards}\}$

Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards **in order**. (uniform probability space).

What is  $P(\text{Ace of Spades First} \cap \text{10 of Clubs Second} \cap \text{4 of Diamonds Third}) = P(A \cap B \cap C)$ ?



**A:** Ace of Spades First

**B:** 10 of Clubs Second

**C:** 4 of Diamonds Third

$$P(A \cap B \cap C) = P(A) P(B|A) P(C|A \cap B)$$

$$P(A) = \frac{51!}{52!} = \frac{1}{52}$$

AS



$P(A \cap B)$

$P(A \cap B \cap C)$

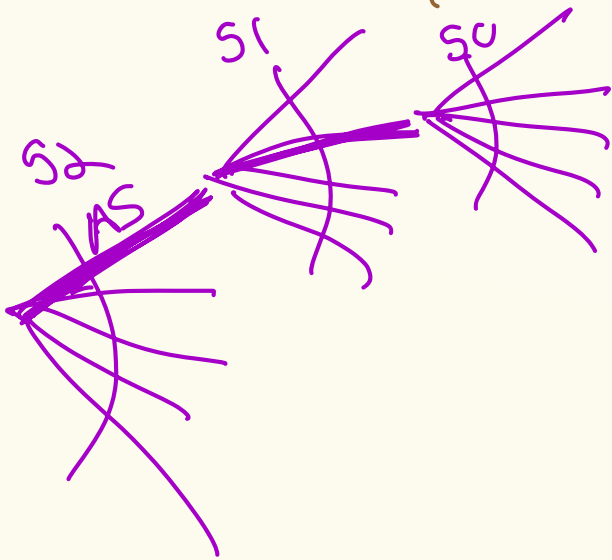
$50!$

$1$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{|A \cap B|}{|A|} = \frac{50!}{51!} = \frac{1}{51}$$

$$P(C|A \cap B) = \frac{|A \cap B \cap C|}{|A \cap B|} = \frac{49!}{50!} = \frac{1}{50}$$

AS 10C 

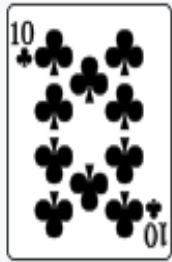


- Step 1: select unif random card.
- Step 2: select unif " " from leftover
- Step 3: select unif from leftover after first 2.

## Chain Rule Example

Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards **in order**. (uniform probability space).

What is  $P(\text{Ace of Spades First, 10 of Clubs Second, 4 of Diamonds Third}) = P(A \cap B \cap C)$ ?



$$\mathbb{P}(A) \cdot \mathbb{P}(B|A) \cdot \mathbb{P}(C|A \cap B)$$

$$= \frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

**A:** Ace of Spades First  
**B:** 10 of Clubs Second  
**C:** 4 of Diamonds Third

## Agenda (3)

- Conditional Probability
- Chain Rule
- Independence
- Law of Total Probability
- Bayes Theorem

# Independent Events

Defn: Events  $E$  and  $F$  are **independent**, written  $E \perp F$ , if:

$$P\{E \cap F\} = P\{E\} \cdot P\{F\}$$

Here's an equivalent and perhaps more intuitive definition:

Defn: Assuming  $P\{F\} > 0$ , Events  $E$  and  $F$  are **independent**, if:

$$P\{E | F\} = P\{E\}$$

$$\begin{aligned} P(E|F) &= \frac{P(E \cap F)}{P(F)} \\ &= \frac{P(E)P(F)}{P(F)} \end{aligned}$$

If  $E$  and  $F$  are **independent**, so are  $\bar{E}$  and  $F$ ,  $E$  and  $\bar{F}$ , and  $\bar{E}$  and  $\bar{F}$

# Independent Events – example 1

Defn: Events  $E$  and  $F$  are **independent**, written  $E \perp F$ , if:

$$P\{E \cap F\} = P\{E\} \cdot P\{F\}$$

Here's an equivalent and perhaps more intuitive definition:

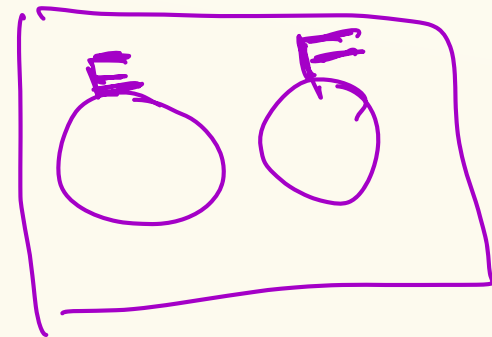
Defn: Assuming  $P\{F\} > 0$ , Events  $E$  and  $F$  are **independent**, if:

$$P\{E | F\} = P\{E\}$$

**Q1**: Can two mutually exclusive, non-null events be independent?

No

Pr > 0



# Independent Events – example 2

Defn: Events  $E$  and  $F$  are **independent**, written  $E \perp F$ , if:

$$P\{E \cap F\} = P\{E\} \cdot P\{F\}$$

Here's an equivalent and perhaps more intuitive definition:

Defn: Assuming  $P\{F\} > 0$ , Events  $E$  and  $F$  are **independent**, if:

$$P\{E | F\} = P\{E\}$$

**Q2**: Suppose we roll a die twice. Each outcome equally likely.  
Which of these pairs of events are independent:

- a. Let  $E$  = "1<sup>st</sup> roll is 6." Let  $F$  = "2<sup>nd</sup> roll is 6"
- b. Let  $E$  = "Sum of rolls is 7." Let  $F$  = "2<sup>nd</sup> roll is 4"

} yes.

$$|\Omega| = 36$$

$$a) P(E) = \frac{6}{36} = \frac{1}{6} \quad P(F) = \frac{1}{6}$$

$$P(E \cap F) = P((6,6)) = \frac{1}{36}$$

$$b) E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$P(E) = \frac{6}{36} = \frac{1}{6} \quad P(F) = \frac{1}{6}$$

$$P(E \cap F) = \frac{1}{36}$$

# Independent Events - answers

Defn: Events  $E$  and  $F$  are **independent**, written  $E \perp F$ , if:

$$P\{E \cap F\} = P\{E\} \cdot P\{F\}$$

Here's an equivalent and perhaps more intuitive definition:

Defn: Assuming  $P\{F\} > 0$ , Events  $E$  and  $F$  are **independent**, if:

$$P\{E \mid F\} = P\{E\}$$

**Q1**: Can two mutually exclusive, non-null events be independent?

No!

**Q2**: Suppose we roll a die twice. Which of these pairs of events are independent:

a. Let  $E$  = "1<sup>st</sup> roll is 6." Let  $F$  = "2<sup>nd</sup> roll is 6"

Both!

b. Let  $E$  = "Sum of rolls is 7." Let  $F$  = "2<sup>nd</sup> roll is 4"