

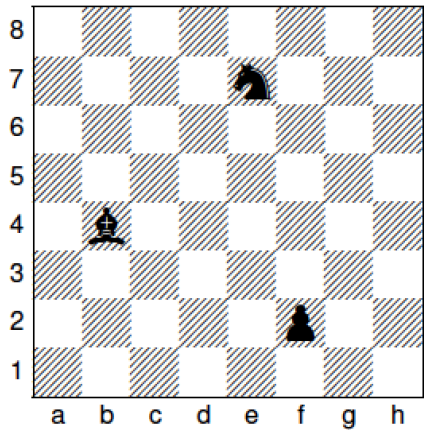
Intro to discrete probability

CSE 312 Spring 26
Lecture 4

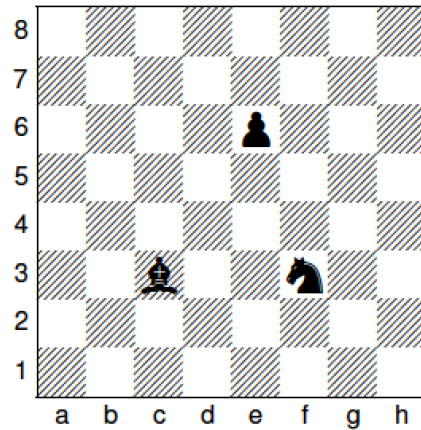
Several slides in this presentation were created by Mor Harchol-Balter in conjunction with her book "Introduction to Probability for Computing", Harchol-Balter '24

8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column?



(a) valid



(b) invalid

Poll:

A. $\binom{64}{3}$

B. $\binom{8}{3} \cdot \binom{8}{3}$

C. $8^2 \cdot 7^2 \cdot 6^2$

D. I don't know.

Possible answers

$$\binom{64}{3}$$

Choose 3 positions on board

$$\binom{8}{3} \cdot \binom{8}{3}$$

Choose 3 rows, then choose 3 cols

$$8^2 \cdot 7^2 \cdot 6^2$$

Choose row/col for pawn, then row/col for bishop, then row/col for knight.

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Probability (intro – 1)

- We want to model uncertainty.
 - i.e., outcome not determined a-priori
 - E.g. throwing dice, flipping a coin...
 - We want to numerically measure likelihood of outcomes = probability.
 - We want to make complex statements about these likelihoods.

Probability (intro – 2)

- We will not argue why a certain physical process realizes the probabilistic model we study
 - Why is the outcome of the coin flip really “random”?
- First part of class: “Discrete” probability theory
 - Experiment with finite / discrete set of outcomes.
 - Will explore countably infinite and continuous outcomes later

Agenda

- Sample Space and Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Sample Space

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Events (mutual exclusivity)

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Definition. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$ (i.e., can't happen at same time)

Examples:

- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$

Example: rolling two dice

Probability is defined in terms of some experiment.

Ω = Sample space of the experiment = Set of all possible outcomes

Defn: An **event**, E , is any subset of the sample space, Ω .

Example: Roll die twice



Q: What does event E_1 represent?

Q: What is $E_1 \cup E_2$?

Q: What is $\overline{E_1}$?

	E_1		E_2		
$\Omega =$	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Sample space and events

Defn: If $E_1 \cap E_2 = \emptyset$, then E_1 and E_2 are **mutually exclusive**.

Defn: If E_1, E_2, \dots, E_n are events such that $E_i \cap E_j = \emptyset, \forall i \neq j$, and such that $\bigcup_{i=1}^n E_i = F$ then we say that events E_1, E_2, \dots, E_n **partition** set F .

Q1: Do E_1 and E_2 partition Ω ?

Q2: What is an example of events that partition Ω for 2 rolls of a die?

	E_1		E_2		
$\Omega =$	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
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Agenda (Part 2)

- Sample Space and Events
- **Probability**
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

$$\mathbb{P}: \Omega \rightarrow [0, 1]$$

that maps outcomes $\omega \in \Omega$ to probabilities.

– Also use notation: $\mathbb{P}(\omega) = P(\omega) = \text{Pr}(\omega)$

Example – Coin Tossing

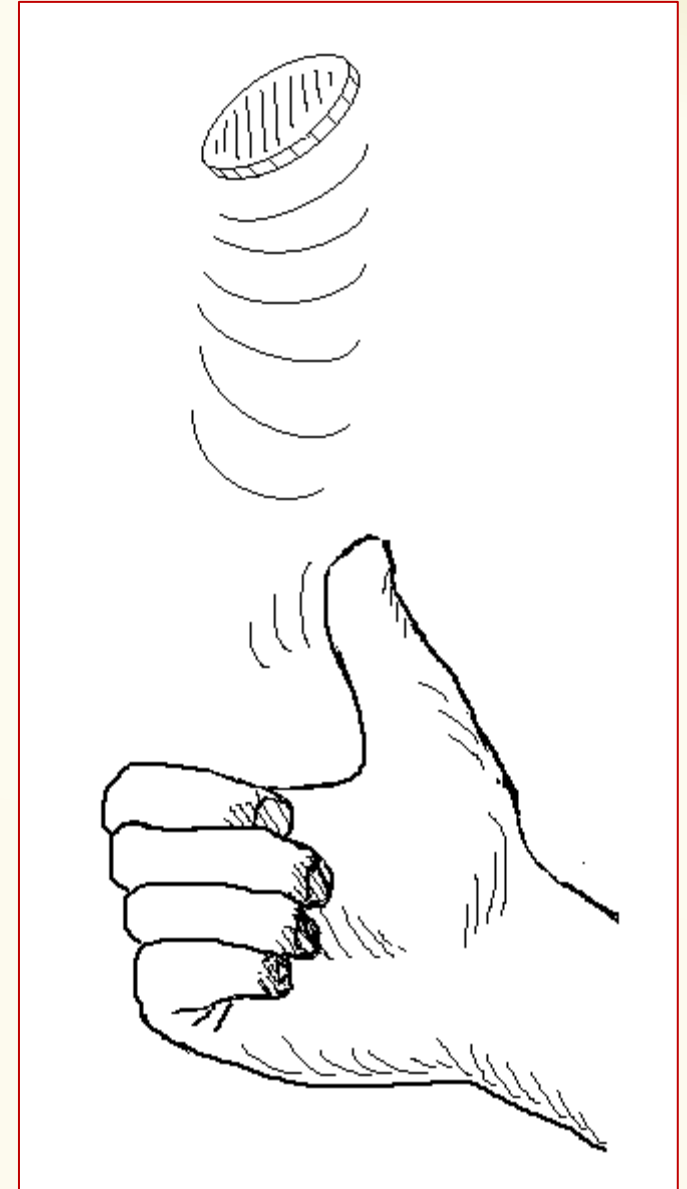
Imagine we toss one coin – outcome can be **heads** or **tails**.

$$\Omega = \{H, T\}$$

\mathbb{P} ? Depends! What do we want to model?!

Fair coin toss

$$\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2} = 0.5$$



Example – Coin Tossing – biased coin

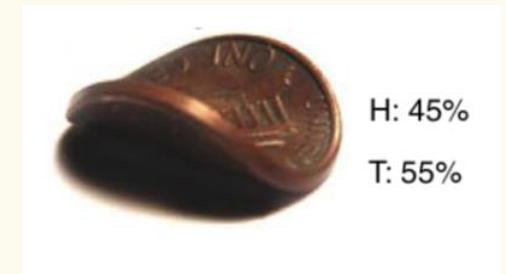
Imagine we toss one coin – outcome can be **heads** or **tails**.

$$\Omega = \{H, T\}$$

\mathbb{P} ? **Depends! What do we want to model?!**

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(H) = 0.45, \quad \mathbb{P}(T) = 0.55$$



Probability space

Definition. A (discrete) **probability space** is a pair (Ω, \mathbb{P}) where:

- Ω is a set called the **sample space**.
- \mathbb{P} is the **probability measure**,

a function $\mathbb{P}: \Omega \rightarrow [0,1]$ such that:

- $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
- $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Probability space - more

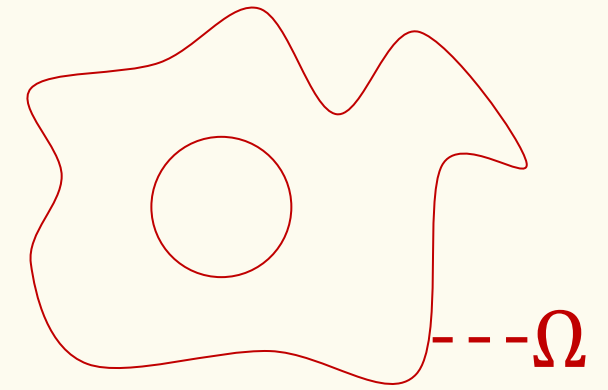
Either finite or infinite countable (e.g., integers)

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Set of possible elementary outcomes



Specify Likelihood (or probability) of each elementary outcome

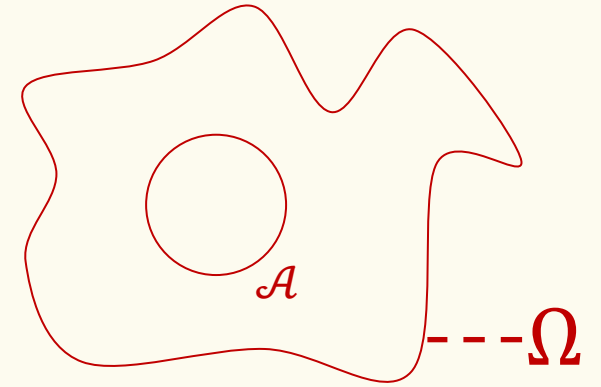
Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Events (more formally)

Definition. An **event** in a probability space (Ω, \mathbb{P}) is a subset $\mathcal{A} \subseteq \Omega$. Its probability is

$$\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)$$



Convenient abuse of notation: \mathbb{P} is extended to be defined over **sets**. $\mathbb{P}(\omega) = \mathbb{P}(\{\omega\})$

Agenda (Part 3)

- Sample Space and Events
- Probability
- **Equally Likely Outcomes**
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Uniform Probability Space = Equally Likely Outcomes

Definition. A uniform probability space is a pair (Ω, \mathbb{P}) such that

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

for all $\omega \in \Omega$.

Examples:

- Fair coin $P(\omega) = \frac{1}{2}$
- Fair 6-sided die $P(\omega) = \frac{1}{6}$

Uniform Probability Space - Corollary

Definition. A **uniform probability space** is a pair (Ω, \mathbb{P})

such that $\mathbb{P}(\omega) = \frac{1}{|\Omega|}$ for all $\omega \in \Omega$.

Definition. An **event** in a probability space (Ω, \mathbb{P}) is a subset $\mathcal{A} \subseteq \Omega$.

Its probability is $\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)$

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$, then

$$P(E) = \frac{|E|}{|\Omega|}$$

Example: rolling two dice (probabilities)

Back to the example of rolling a die twice.

Suppose that each outcome is equally likely

What is the size of the sample space Ω ?

Example: Roll die twice



Q: What is the probability of event E_1 ?

Q: What is probability of event $\overline{E_1}$?

Q: What is the probability of event $E_1 \cup E_2$?

	E_1			E_2	
$\Omega =$	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
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	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Example – Coin Tossing question

Toss a coin 100 times. Each outcome is **equally likely**. What is the probability of seeing 50 heads?

<https://pollev.com/annakarlin185>

(A) $\frac{1}{2}$

(B) $\frac{1}{2^{50}}$

(C) $\frac{\binom{100}{50}}{2^{100}}$

(D) Not sure

Agenda (Part 4)

- Sample Space and Events
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- Equally Likely Outcomes
- **Probability Axioms and Beyond Equally Likely Outcomes**
- More Examples

Axioms of Probability

$P\{E\}$ = probability of event E
= probability that the outcome of the experiment lies in set E

The 3 Probability Axioms:

Non-negativity: $P\{E\} \geq 0$ for any event E .

Additivity: If E_1, E_2, E_3, \dots is a countable sequence of disjoint events, then

$$P\{E_1 \cup E_2 \cup E_3 \cup \dots\} = P\{E_1\} + P\{E_2\} + P\{E_3\} + \dots$$

Normalization: $P\{\Omega\} = 1$

Corollaries of Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this applies to **any** probability space (not just uniform)

Axiom 1 (Non-negativity): $P(E) \geq 0$.

Axiom 2 (Normalization): $P(\Omega) = 1$

Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

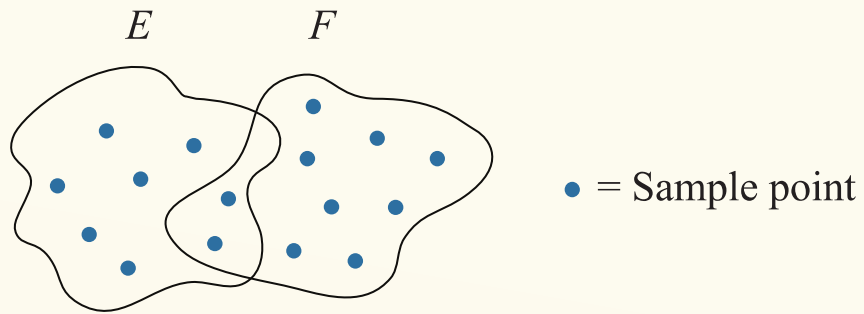
Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.

Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$

Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Inclusion-exclusion

Lemma: $P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\}$



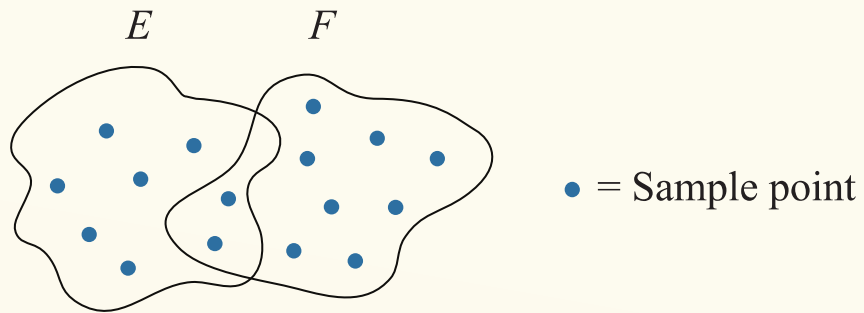
Proof: (Hint: Think about Additivity Axiom)

Express $E \cup F$ as a union of mutually exclusive sets

$$E \cup F = E \cup (F \setminus (E \cap F))$$

Inclusion- exclusion continued

Lemma: $P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\}$



Proof:

Express $E \cup F$ as a union of mutually exclusive sets

$$E \cup F = E \cup (F \setminus (E \cap F))$$

Then, by the Additivity Axiom we have 2 observations:

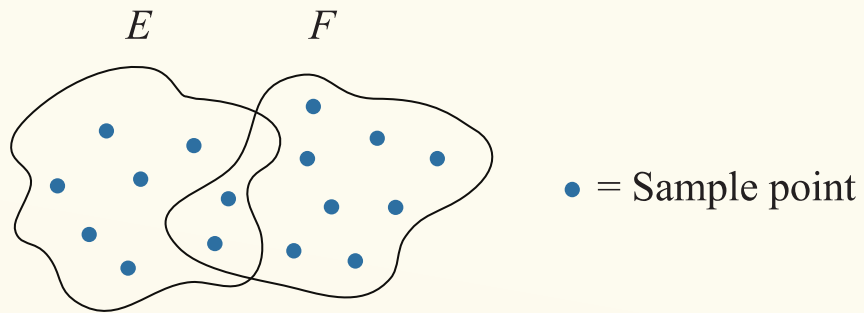
$$P\{E \cup F\} = P\{E\} + P\{F \setminus (E \cap F)\}$$

$$P\{F\} = P\{F \setminus (E \cap F)\} + P\{E \cap F\}$$

Now substitute the 2nd equation into the 1st . ■

Corollary

Lemma: $\mathbf{P}\{E \cup F\} = \mathbf{P}\{E\} + \mathbf{P}\{F\} - \mathbf{P}\{E \cap F\}$



Proof:

Express $E \cup F$ as a union of mutually exclusive sets

$$E \cup F = E \cup (F \setminus (E \cap F))$$

Then, by the Additivity Axiom we have 2 observations:

$$\mathbf{P}\{E \cup F\} = \mathbf{P}\{E\} + \mathbf{P}\{F \setminus (E \cap F)\}$$

$$\mathbf{P}\{F\} = \mathbf{P}\{F \setminus (E \cap F)\} + \mathbf{P}\{E \cap F\}$$

Now substitute the 2nd equation into the 1st. ■

Lemma: $\mathbf{P}\{E \cup F\} \leq \mathbf{P}\{E\} + \mathbf{P}\{F\}$

Proof: WHY??

Review: Probability space

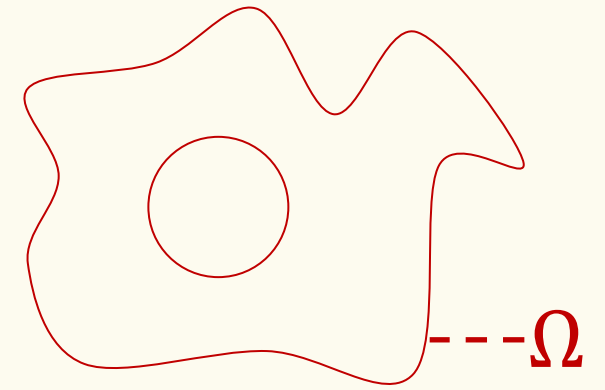
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Set of possible elementary outcomes



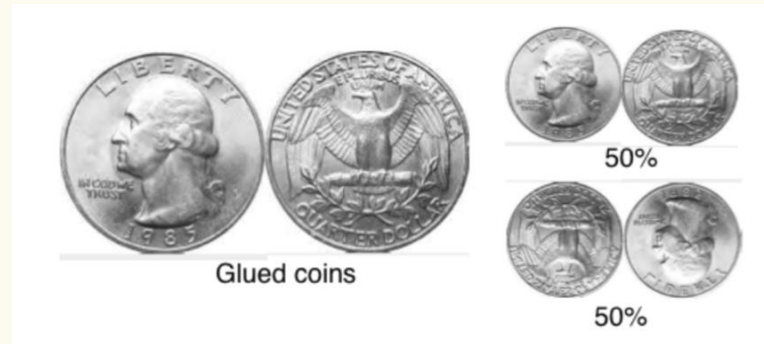
Specify Likelihood (or probability) of each elementary outcome

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Non-equally Likely Outcomes

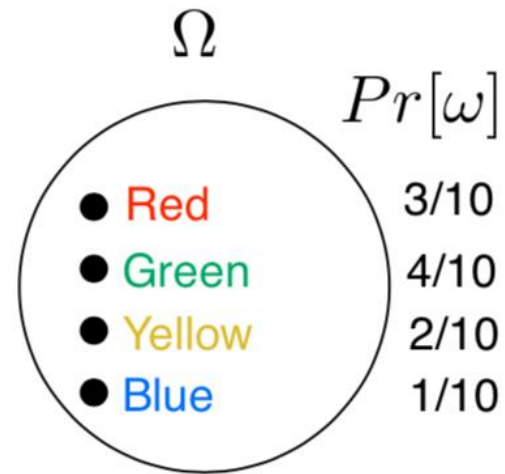
Probability spaces can have **non-equally likely outcomes**.



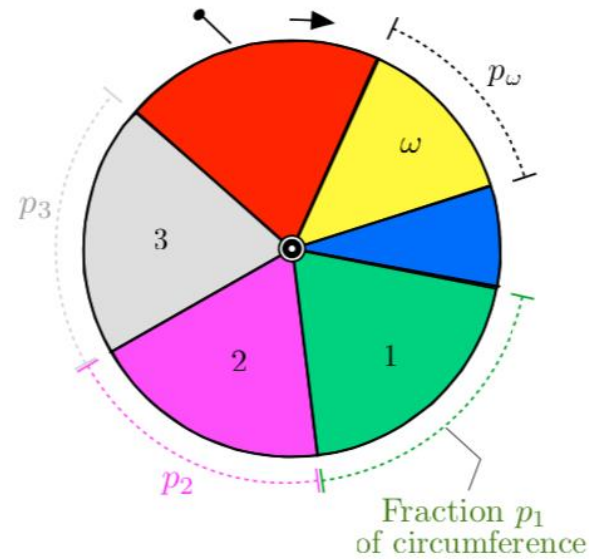
More Examples of Non-equally Likely Outcomes



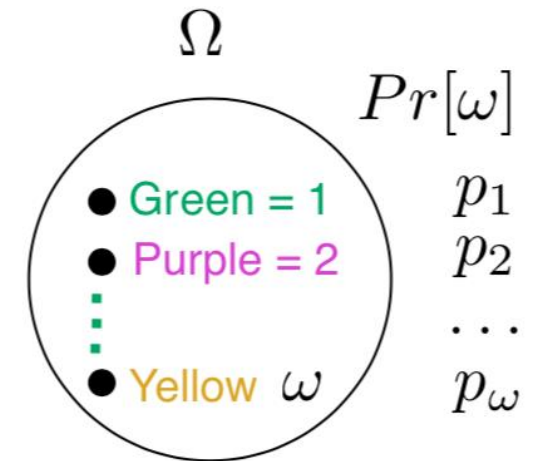
Physical experiment



Probability model



Physical experiment



Probability model

Agenda (Part 5)

- Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- **More Examples**

Example: Dice Rolls

Suppose I had two, fair, 6-sided dice, one green and one red. I roll them. What is the probability that we see at least one 3 in the two rolls?

Size of sample space:

Size of event:

Probability:

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Size of sample space?

<i>Outcomes</i>
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

Example: Returning Homeworks ($n=3$)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- If $n=3$, what is the probability that all 3 people get their own homework back?

<i>Outcomes</i>
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

Example: Returning Homeworks (arbitrary n)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- For arbitrary n , what is the probability that all n people get their own homework back?

<i>Outcomes</i>
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

Example: Returning Homeworks (person 3)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- For $n=3$, what is the probability that person 3 gets their own homework back?

<i>Outcomes</i>
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

Example: Returning Homeworks (large n , person 17)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- What is the probability that person 17 gets their own homework back?

<i>Outcomes</i>
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

Example: Birthday “Paradox”

Suppose we have a collection of n people in a room. What is the probability that at least 2 people share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.