Intro to discrete probability

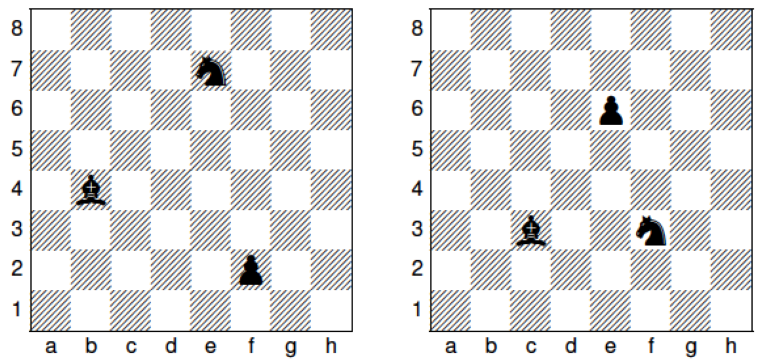
CSE 312 Spring 26

Lecture 4

Several slides in this presentation were created by Mor Harchol-Balter in conjunction with her book "Introduction to Probability for Computing", Harchol-Balter '24

8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column ?



(a) valid (b) invalid

Poll:

A. $\binom{64}{3}$

B. $\binom{8}{3} \cdot \binom{8}{3}$

C. $8^2 \cdot 7^2 \cdot 6^2$

D. I don't know.

Possible answers

overcount   undercount

$$\binom{64}{3}$$

Choose 3 positions on board

p b k

r1 r2 r3
c1 c2 c3

$$\binom{8}{3} \cdot \binom{8}{3}$$

Choose 3 rows, then choose 3 cols

$$8^2 \cdot 7^2 \cdot 6^2$$

Choose row/col for pawn, then row/col for bishop, then row/col for knight.

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Probability (intro – 1)

- We want to model uncertainty.
  - i.e., outcome not determined a-priori
  - E.g. throwing dice, flipping a coin…
  - We want to numerically measure likelihood of outcomes = probability.
  - We want to make complex statements about these likelihoods.

Probability (intro – 2)

- We will not argue why a certain physical process realizes the probabilistic model we study
  - Why is the outcome of the coin flip really "random"?

- First part of class: "Discrete" probability theory
  - Experiment with finite / discrete set of outcomes.
  - Will explore countably infinite and continuous outcomes later

Agenda

- Sample Space and Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Sample Space

Omega

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Events (mutual exclusivity)

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Definition. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$ (i.e., can't happen at same time)

Examples:

- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$

Example: rolling two dice

Probability is defined in terms of some experiment.

Ω = Sample space of the experiment = Set of all possible outcomes

$|\Omega| = 36$

<u>Defn</u>: An **event**, E, is any subset of the sample space, Ω.

<u>Example</u>: Roll die twice 

Q: What does event E_1 represent?
Q: What is $E_1 \cup E_2$?
Q: What is $\overline{E_1}$? E_1^c

$\Omega =$

	E_1			E_2	
(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
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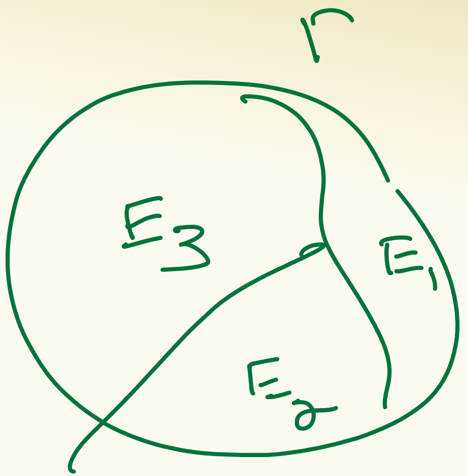
Sample space and events



<u>Defn</u>: If $E_1 \cap E_2 = \emptyset$, then E_1 and E_2 are **mutually exclusive**.

<u>Defn</u>: If $E_1, E_2, \ldots, E_n$ are events such that $E_i \cap E_j = \emptyset, \ \forall i \neq j$, and such that $\bigcup_{i=1}^{n} E_i = F$ then we say that events $E_1, E_2, \ldots, E_n$ **partition** set F.

$\frac{1}{36}$

Q1: Do E_1 and E_2 partition Ω ?

Q2: What is an example of events that partition Ω for 2 rolls of a die?

$$P(E_1) = \frac{6}{36}$$

$$\begin{array}{c} \\ \Omega = \end{array} \left\{ \begin{array}{cccccc} & E_1 & & & E_2 & \\ (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right.$$

Agenda (Part 2)

- Sample Space and Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

$$\mathbb{P}\colon \Omega \to [0,1]$$

that maps outcomes $\omega \in \Omega$ to probabilities.

– Also use notation: $\mathbb{P}(\omega) = P(\omega) = \Pr(\omega)$

Example – Coin Tossing

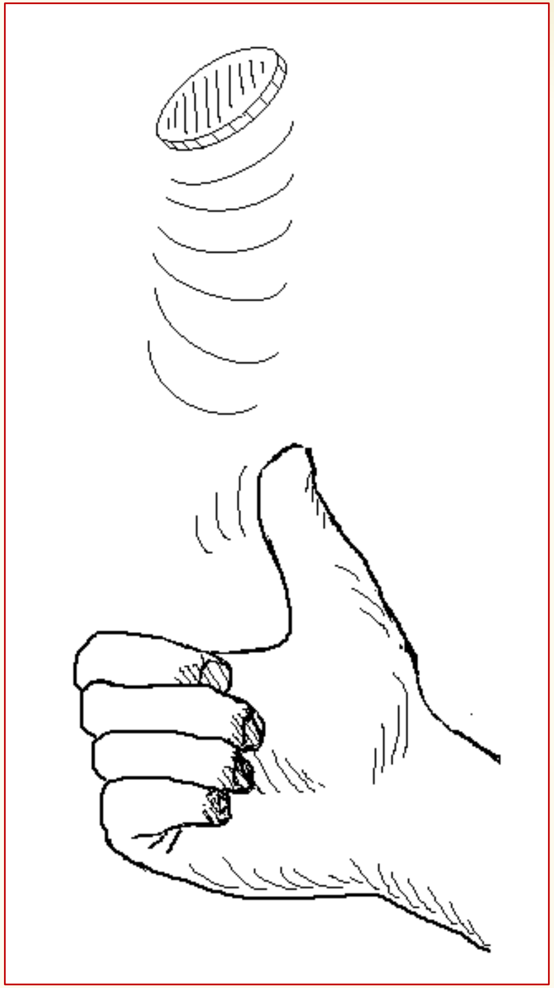
Imagine we toss one coin – outcome can be **heads** or **tails.**

$$\Omega = \{H, T\}$$

$\mathbb{P}$? Depends! What do we want to model?!

Fair coin toss

$$\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2} = 0.5$$



Example – Coin Tossing – biased coin

Imagine we toss <u>one</u> coin – outcome can be **heads** or **tails.**

$$\Omega = \{H, T\}$$

$\mathbb{P}$? Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(H) = 0.45, \qquad \mathbb{P}(T) = 0.55$$

Probability space

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

- Ω is a set called the **sample space**.
- $\mathbb{P}$ is the **probability measure,**

  a function $\mathbb{P}: \Omega \to [0,1]$ such that:
  - $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Probability space - more

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

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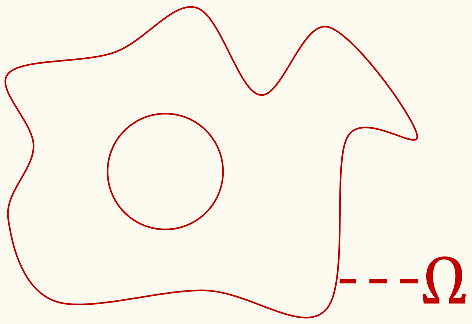
Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

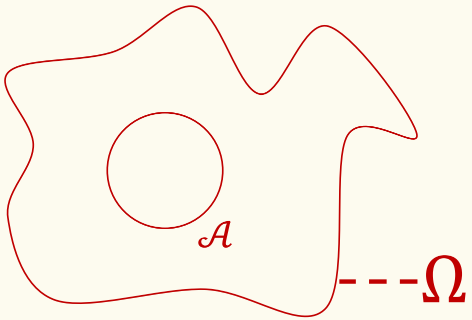
Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Events (more formally)

Definition. An **event** in a probability space $(\Omega, \mathbb{P})$ is a subset $\mathcal{A} \subseteq \Omega$. Its probability is

$$\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)$$



Convenient abuse of notation: $\mathbb{P}$ is extended to be defined over **sets**.

$$\mathbb{P}(\omega) = \mathbb{P}(\{\omega\})$$

Agenda (Part 3)

- Sample Space and Events
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Uniform Probability Space = Equally Likely Outcomes

Definition. A **uniform probability space** is a pair $(\Omega, \mathbb{P})$ such that

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

for all $\omega \in \Omega$.

Examples:

- Fair coin $P(\omega) = \frac{1}{2}$
- Fair 6-sided die $P(\omega) = \frac{1}{6}$

Uniform Probability Space - Corollary

Definition. A **uniform** probability space is a pair $(\Omega, \mathbb{P})$ such that $\mathbb{P}(\omega) = \frac{1}{|\Omega|}$. for all $\omega \in \Omega$.

Definition. An **event** in a probability space $(\Omega, \mathbb{P})$ is a subset $\mathcal{A} \subseteq \Omega$.
Its probability is $\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)$

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$, then

$$P(E) = \frac{|E|}{|\Omega|}$$

$$P(E) = \sum_{\omega \in E} P(\omega), \quad P(\omega) = \frac{1}{|\Omega|}$$

Example: rolling two dice (probabilities)

Back to the example of rolling a die twice.

Suppose that each outcome is equally likely

<u>What is the size of the sample space</u> Ω?

$P(E_1) = \frac{6}{36} = \frac{1}{6}$

$P(\overline{E_1}) = \frac{30}{36} = \frac{5}{6}$

$P(E_1 \cup E_2) = \frac{9}{36} = \frac{1}{4}$

$|\Omega| = 36$

<u>Example</u>: Roll die twice

Q: What is the probability of event E_1?
Q: What is probability of event $\overline{E_1}$?
Q: What is the probability of event $E_1 \cup E_2$?

$\Omega =$

	E_1			E_2	
(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
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Example – Coin Tossing question

$\Omega = \{\text{all possible seqs of 100 H/T}\}$ $\quad |\Omega| = 2^{100}$

Toss a coin 100 times. Each outcome is **equally likely**. What is the probability of seeing 50 heads?

https://pollev.com/ annakarlin185

$Pr(\text{see 50 H}) = \frac{\#\text{seqs w/ 50 H}}{|\Omega|}$, where $\#\text{seqs w/ 50 H} = \binom{100}{50}$

(A) $\frac{1}{2}$

(B) $\frac{1}{2^{50}}$

(C) $\frac{\binom{100}{50}}{2^{100}}$

(D) Not sure

Agenda (Part 4)

- Sample Space and Events
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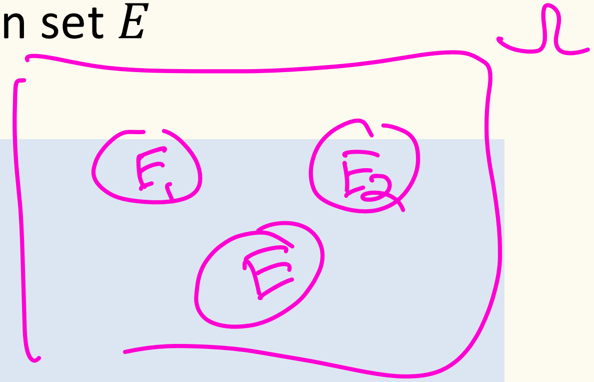
Axioms of Probability

$\boldsymbol{P}\{E\}$ = probability of event E
= probability that the outcome of the experiment lies in set E



The 3 Probability Axioms:

Non-negativity: $\boldsymbol{P}\{E\} \geq 0$ for any event E.

Additivity: If $E_1, E_2, E_3, \ldots$ is a countable sequence of disjoint events, then

$$\boldsymbol{P}\{E_1 \cup E_2 \cup E_3 \cup \cdots\} = \boldsymbol{P}\{E_1\} + \boldsymbol{P}\{E_2\} + \boldsymbol{P}\{E_3\} + \cdots$$

Normalization: $\boldsymbol{P}\{\Omega\} = 1$

Corollaries of Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is applies to **any** probability space (not just uniform)

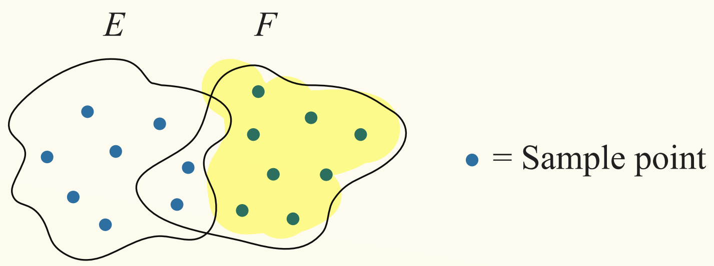
Axiom 1 (Non-negativity): $P(E) \geq 0.$
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

$$1 = P(\Omega) = P(E \cup E^c) = P(E) + P(E^c)$$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E).$
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Inclusion-exclusion

Lemma: $\boldsymbol{P}\{E \cup F\} = \boldsymbol{P}\{E\} + \boldsymbol{P}\{F\} - \boldsymbol{P}\{E \cap F\}$



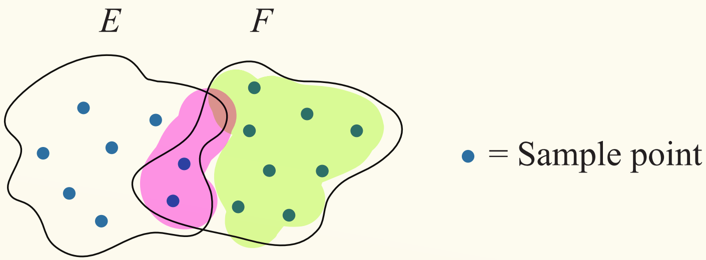
Proof: (Hint: Think about Additivity Axiom)

Express $E \cup F$ as a union of mutually exclusive sets

$$E \cup F = E \cup \big(F \setminus (E \cap F)\big)$$

Inclusion- exclusion continued

Lemma: $\boldsymbol{P}\{E \cup F\} = \boldsymbol{P}\{E\} + \boldsymbol{P}\{F\} - \boldsymbol{P}\{E \cap F\}$



Proof:

Express $E \cup F$ as a union of mutually exclusive sets

$$E \cup F = E \cup \big(F \setminus (E \cap F)\big)$$

Then, by the Additivity Axiom we have 2 observations:

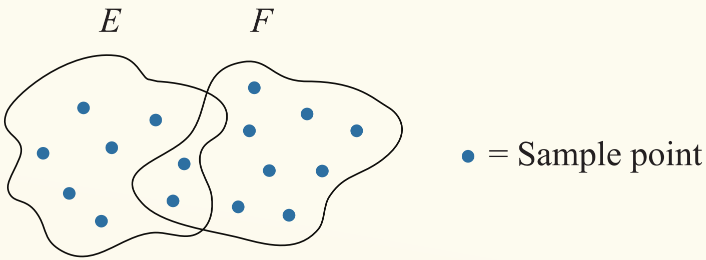
$$\boldsymbol{P}\{E \cup F\} = \mathbf{P}\{E\} + \boldsymbol{P}\{F \setminus (E \cap F)\}$$

$$\boldsymbol{P}\{F\} = \boldsymbol{P}\{F \setminus (E \cap F)\} + \boldsymbol{P}\{E \cap F\}$$

Now substitute the 2nd equation into the 1st . ∎

Corollary

Lemma: $\boldsymbol{P}\{E \cup F\} = \boldsymbol{P}\{E\} + \boldsymbol{P}\{F\} - \boldsymbol{P}\{E \cap F\}$



Proof:

Express $E \cup F$ as a union of mutually exclusive sets

$$E \cup F = E \cup \left(F \setminus (E \cap F)\right)$$

Then, by the Additivity Axiom we have 2 observations:

$$\boldsymbol{P}\{E \cup F\} = \mathbf{P}\{E\} + \boldsymbol{P}\{F \setminus (E \cap F)\}$$
$$\boldsymbol{P}\{F\} = \boldsymbol{P}\{F \setminus (E \cap F)\} + \boldsymbol{P}\{E \cap F\}$$

Now substitute the 2^{nd} equation into the 1^{st}. ∎

Lemma: $\boldsymbol{P}\{E \cup F\} \leq \boldsymbol{P}\{E\} + \boldsymbol{P}\{F\}$

Proof: WHY??

Review: Probability space

Either finite or infinite countable (e.g., integers)

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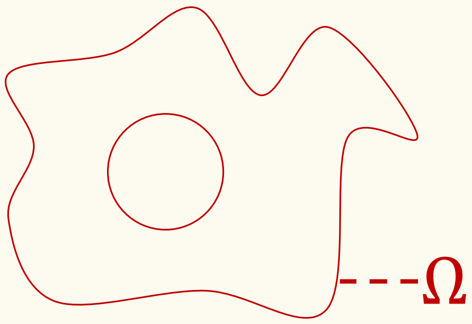
  - $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Set of possible **elementary outcomes**



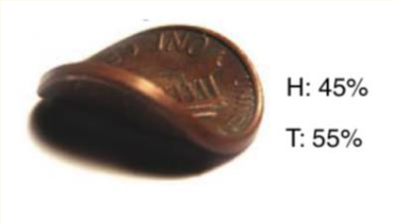
Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Non-equally Likely Outcomes

Probability spaces can have **non-equally likely outcomes.**

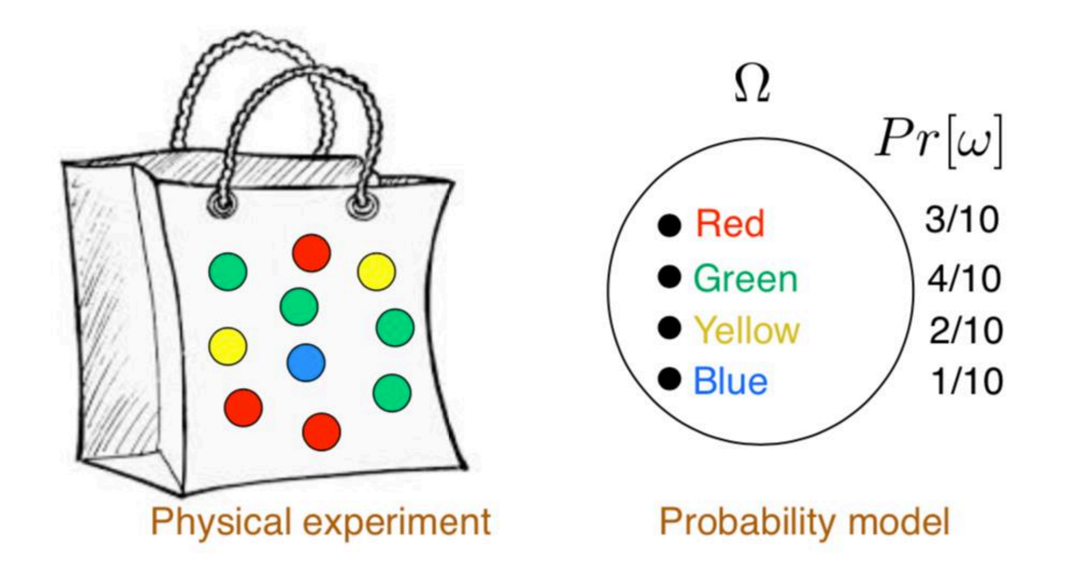


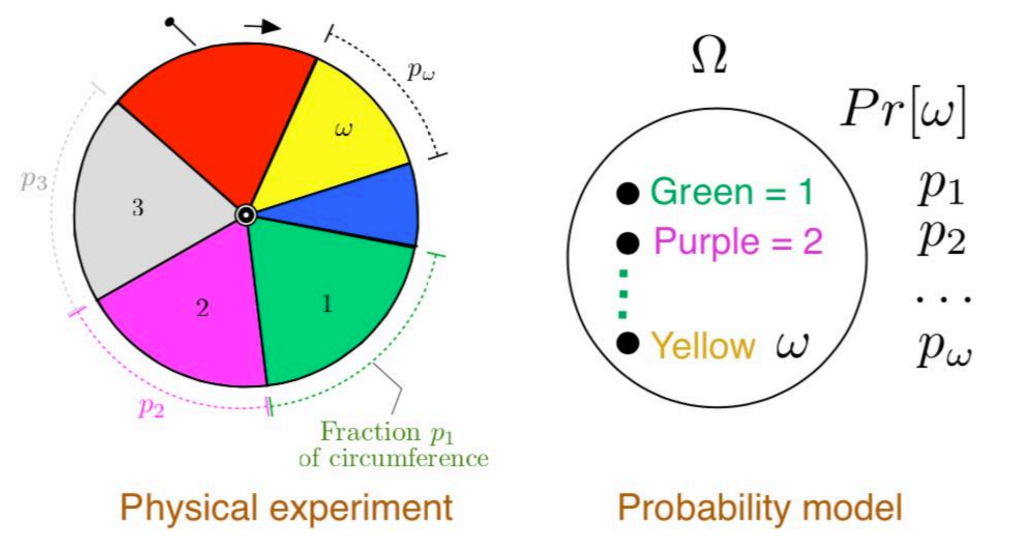
{HH, HT, TH, TT}

0, 1/2, 1/2, 0

0.4, 0.1, 0.1, 0.4

More Examples of Non-equally Likely Outcomes





Agenda (Part 5)

- Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Example: Dice Rolls

equally likely outcomes.

Suppose I had two, fair, 6-sided dice, one green and one red. I roll them. What is the probability that we see at least one 3 in the two rolls?

$P(\omega) = \frac{1}{36}$

Size of sample space: $|\Omega| = 36.$

Size of event: $E = \{\text{at least one } 3\}$

Probability: $P(E) = \frac{|E|}{|\Omega|} = 1 - \frac{|E^c|}{|\Omega|} = 1 - \frac{25}{36}$

$E^c = \{\text{no } 3\text{'s}\}$

$|E^c| = 5^2$

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Size of sample space? $n!$

$n=3$

$s_1 \; s_2 \; s_3$

Outcomes
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

$$Pr(w) = \frac{1}{n!}$$

Example: Returning Homeworks (n=3)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- If n=3, what is the probability that all 3 people get their own homework back?

$\frac{1}{3!}$

Outcomes
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

Example: Returning Homeworks (arbitrary n)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- For arbitrary n, what is the probability that all n people get their own homework back?

Outcomes
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

$$\frac{1}{n!}$$

Example: Returning Homeworks (person 3)

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- For n=3, what is the probability that person 3 gets their own homework back?

Outcomes
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

$\frac{2}{3!}$

Example: Returning Homeworks (large n, person 17

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- What is the probability that person 17 gets their own homework back?

n people.

Outcomes
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

$$\Pr(\quad) = \frac{(n-1)!}{n!} = \frac{1}{n}$$