

# More Counting!

We will use  
[PollEv.com/annakarlin185](https://poll-ev.com/annakarlin185)  
Today again

CSE 312 Spring 26  
Lecture 3

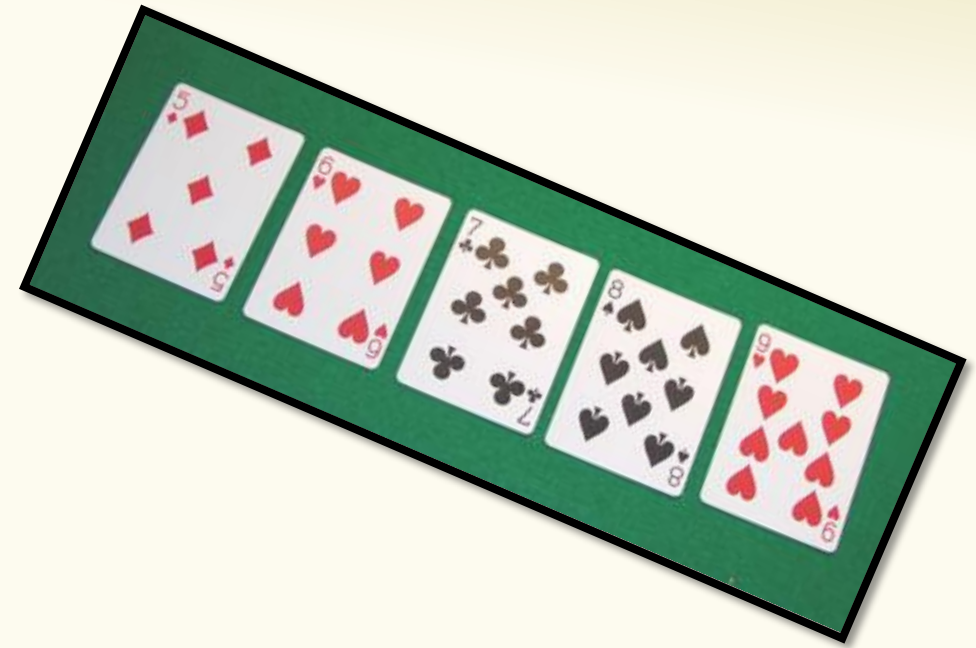
# Recap

- Sum Rule, Product Rule
- Combinations
- Binomial Theorem
- Multinomial Coefficients
- Combinatorial Proofs
- Inclusion-Exclusion

# Today's Agenda (1)

- Sleuth's Criterion
- Stars and Bars
- Pigeonhole Principle
- More examples?

# Quick Review of Cards



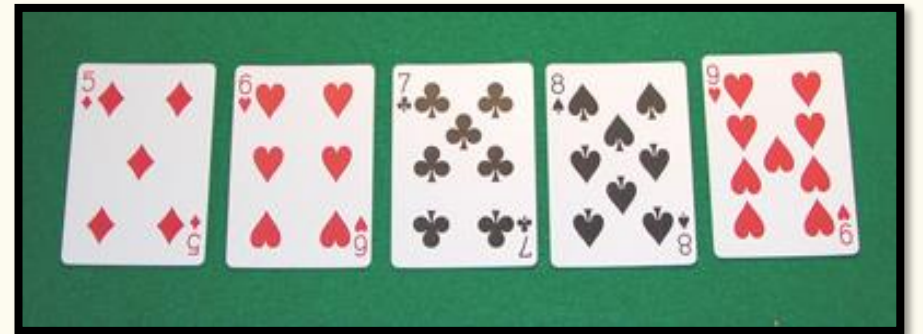
How many possible 5 card hands?

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

# Counting Cards I

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A **straight** is five consecutive rank cards of any suit. How many possible straights?



## Counting Cards II

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

A **flush** is a five card hand all of the same suit.

How many possible flushes?



## Counting Cards III

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

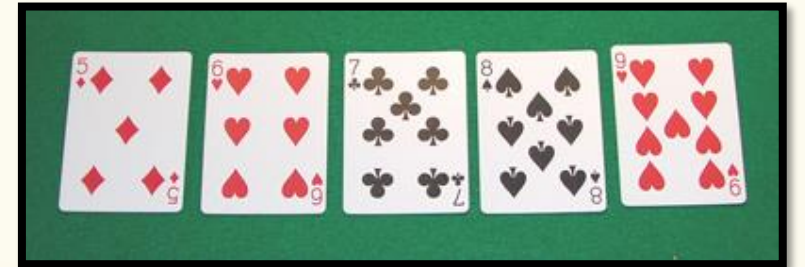
A **flush** is five card hand all of the same suit.

How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$



How many flushes are **NOT** straights?



# Counting Cards IV

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

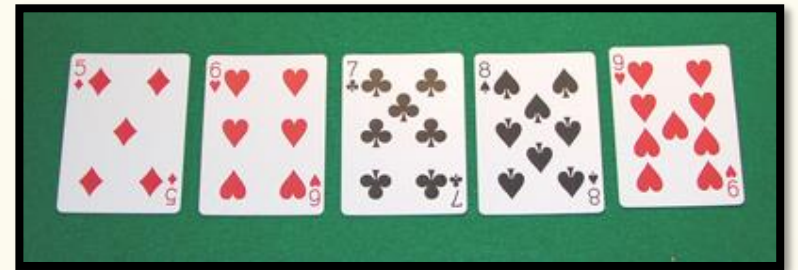
- A flush is five card hand all of the same suit.  
How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$



- How many flushes are **NOT** straights?

= #flush - #flush and straight



$$\left( 4 \cdot \binom{13}{5} = 5148 \right) - 10 \cdot 4$$

## Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence → under counting

Many sequences → over counting

## Sleuth's Criterion (Rudich) - example

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence → under counting      Many sequences → over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.  $\binom{4}{3} \cdot \binom{49}{2}$

## Sleuth's Criterion (Rudich) – poll on example

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence → under counting      Many sequences → over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

1. First choose 3 Aces.

$$\binom{4}{3} \cdot \binom{49}{2}$$

2. Then choose remaining two cards.

Poll:

A. Correct

B. Overcount

C. Undercount

## Sleuth's Criterion (Rudich) – example – where is the error?

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence → under counting      Many sequences → over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

1. First choose 3 Aces.

$$\binom{4}{3} \cdot \binom{49}{2}$$

2. Then choose remaining two cards.

## Sleuth's Criterion (Rudich) - a different way

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence → under counting      Many sequences → over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

When in doubt, break up into disjoint sets you know how to count, and then use the sum rule.

## Sleuth's Criterion (Rudich) – correct solution

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence → under counting      Many sequences → over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

= # 5 card hand containing exactly 3 Aces

+ # 5 card hand containing exactly 4 Aces

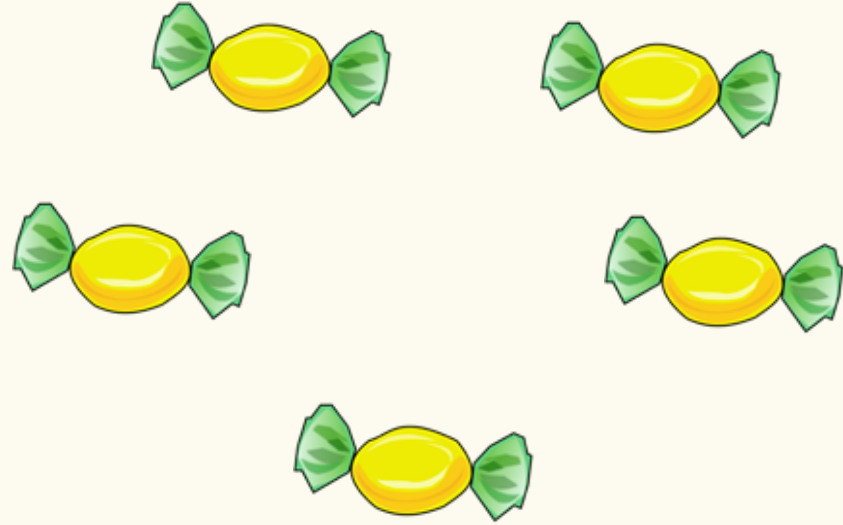
$$\binom{4}{3} \cdot \binom{48}{2}$$

$$\binom{48}{1}$$

# Agenda (1)

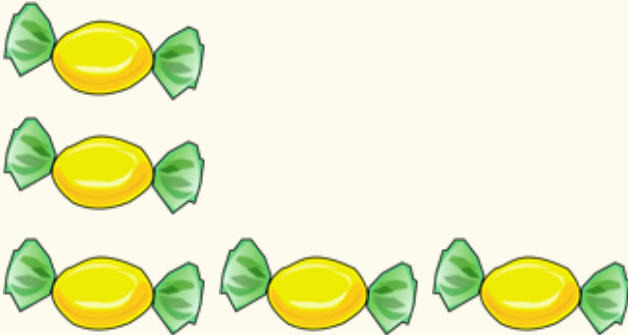
- Sleuth's Criterion
- Stars and Bars
- Pigeonhole Principle
- More examples?

# Example: Kids and Candies



How many ways can we give five **indistinguishable** candies to these three kids?

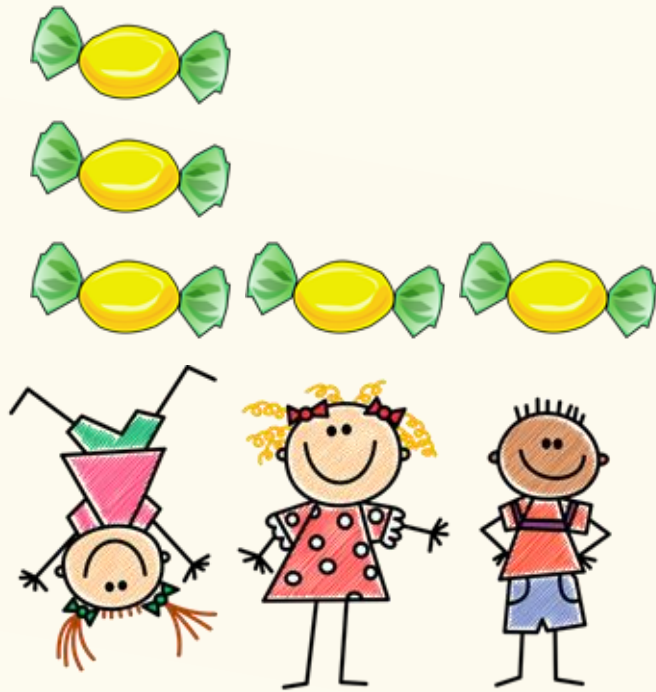
# Kids + Candies



# Kids + Candies



- Idea: count something different

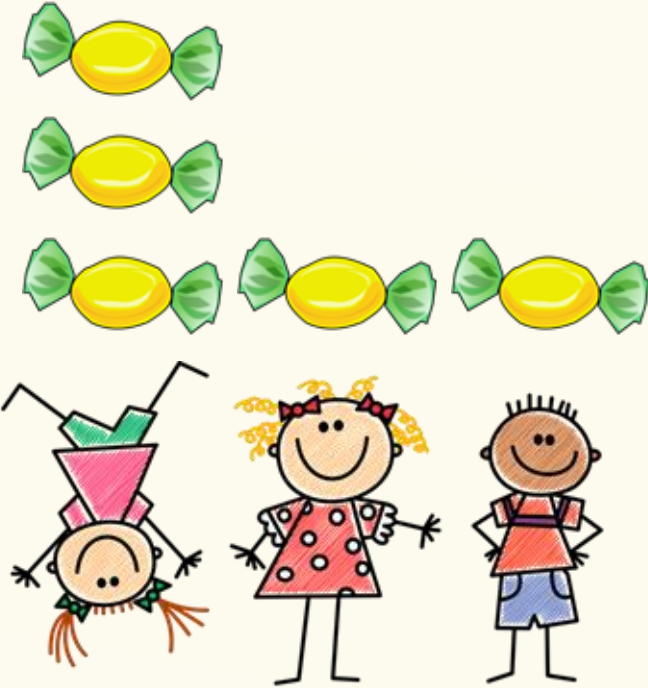




# Kids + Candies: count something equivalent

Idea: Count something equivalent

5 “stars” for candies, 2 “bars” for dividers.

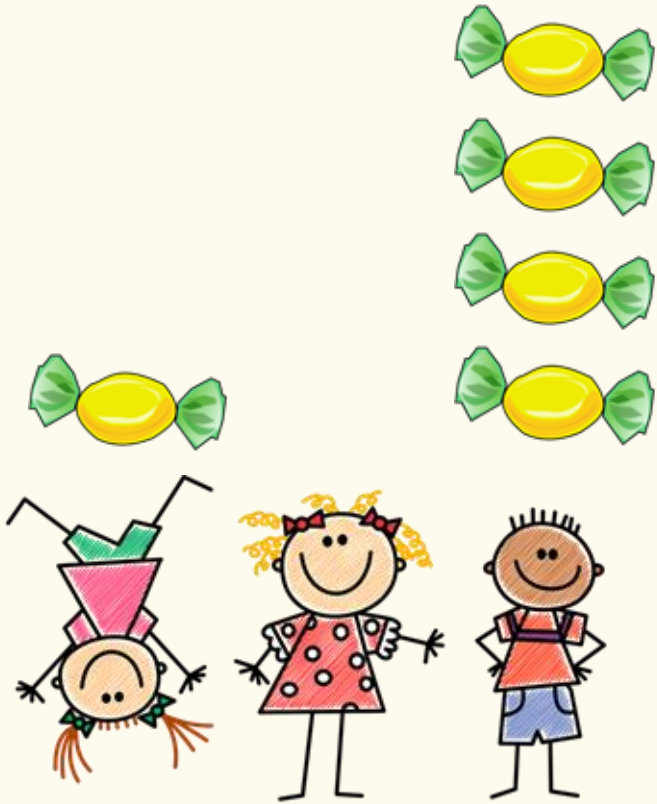




# Kids + Candies: another example

Idea: Count something equivalent

5 “stars” for candies, 2 “bars” for dividers.





# Kids + Candies: bijection

For each candy distribution, there is exactly one corresponding way to arrange the stars and bars.

Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.



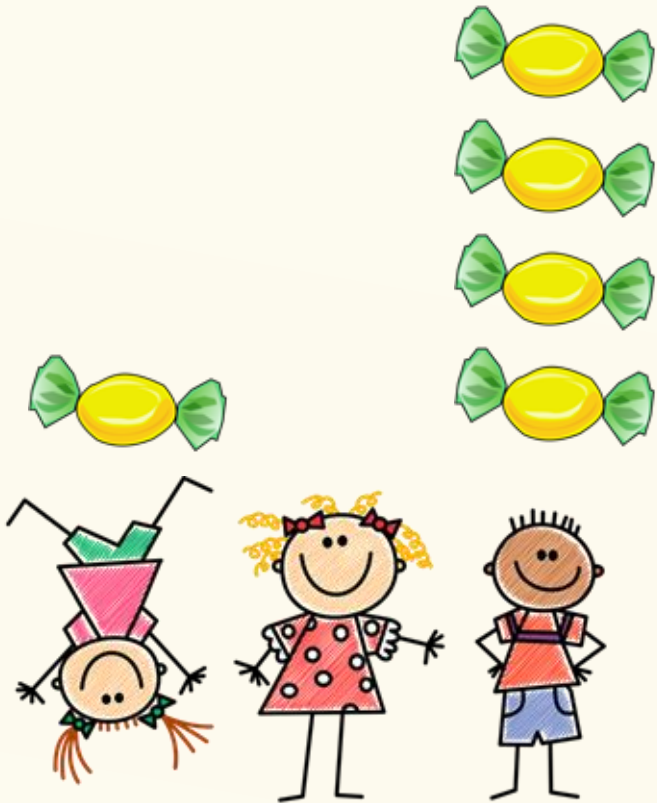


## Kids + Candies: conclusion

Hence, the number of ways to distribute candies to the 3 kids is the number of arrangements of stars and bars.

This is

$$\binom{7}{2} = \binom{7}{5}$$



## Stars and Bars / Divider method

The number of ways to distribute  $n$  indistinguishable balls into  $k$  distinguishable bins is

$$\binom{n + k - 1}{k - 1} = \binom{n + k - 1}{n}$$

## Agenda (3)

- Sleuth's Criterion
- Stars and Bars
- Pigeonhole Principle
- More examples?

# Pigeonhole Principle (PHP): Idea

10 pigeons into 9 pigeonholes

There must be at least one pigeonhole with at least two pigeons



# Pigeonhole Principle

If there are  $n$  pigeons in  $n - 1$  holes or fewer, then one hole must contain **at least 2 pigeons!**

## Pigeonhole Principle – Example

*In a room with 367 people, there are at least two with the same birthday.*

## Pigeonhole Principle – example repeated

*In a room with 367 people, there are at least two with the same birthday.*

Solution:

1. **367** pigeons = people
2. **366** holes = possible birthdays (including leap day)
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

## Pigeonhole Principle – More generally

If there are  $n$  pigeons in  $k < n$  holes, then one hole must contain at least  $n/k$  pigeons!

**Proof.** Assume there are  $< n/k$  pigeons per hole.

Then, there are  $< k(n/k) = n$  pigeons overall.

Contradiction!

## Pigeonhole Principle – Better version

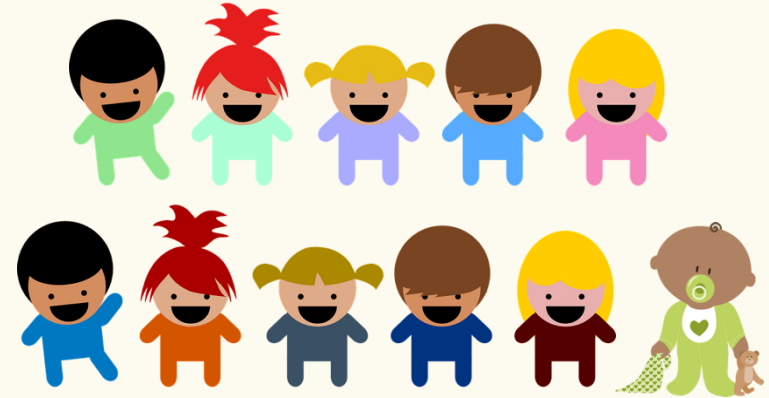
If there are  $n$  pigeons in  $k < n$  holes, then one hole must contain at least  $\left\lceil \frac{n}{k} \right\rceil$  pigeons!

**Reason.** Can't have fractional number of pigeons

Syntax reminder:

- Ceiling:  $\lceil x \rceil$  is  $x$  rounded up to the nearest integer (e.g.,  $\lceil 2.731 \rceil = 3$ )
- Floor:  $\lfloor x \rfloor$  is  $x$  rounded down to the nearest integer (e.g.,  $\lfloor 2.731 \rfloor = 2$ )

## Pigeonhole Principle: Example 2



How many children must share a cake in the worst case?

1. Pigeons: 11 children
2. Pigeonholes:  $k=3$  cakes.
3. Kid goes to cake they will be sharing.
4. By PHP, there will be one cake shared by at least  $\lceil 11/3 \rceil = 4$  kids.

## Pigeonhole Principle – Example 3

*In every set  $S$  of 100 integers, there are at least **two** elements whose difference is a multiple of 37.*

## Pigeonhole Principle – Example 3 - setup

*In every set  $S$  of 100 integers, there are at least **two** elements whose difference is a multiple of 37.*

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

## Pigeonhole Principle – Example 3 (Surprising?)

*In every set  $S$  of 100 integers, there are at least **two** elements whose difference is a multiple of 37.*

When solving a PHP problem:

1. Identify pigeons                      integers in set  $S$
2. Identify pigeonholes                value mod 37
3. Specify how pigeons are  
assigned to pigeonholes               $x$  to hole  $x \bmod 37$
4. Apply PHP                              100 pigeons to 37 PHs

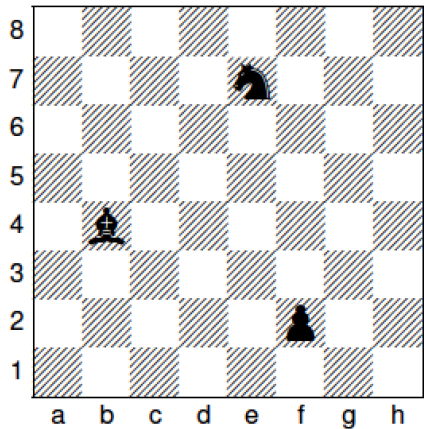
## Agenda (4)

- Sleuth's Criterion
- Stars and Bars
- Pigeonhole Principle
- **More examples?**

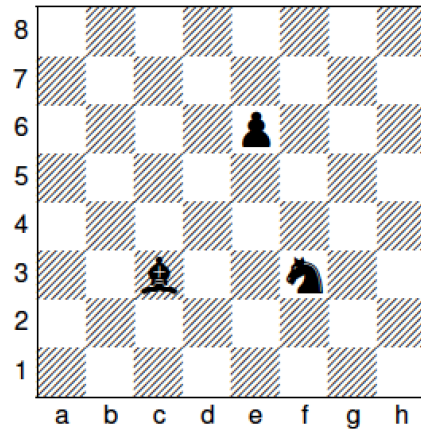


## 8 by 8 chessboard - solution

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column?



(a) valid



(b) invalid

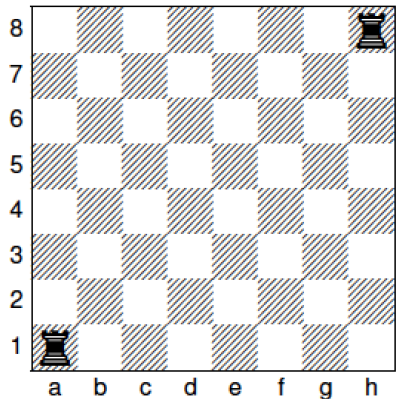
### Sequential process:

1. Column for pawn
2. Row for pawn
3. Column for bishop
4. Row for bishop
5. Column for knight
6. Row for knight

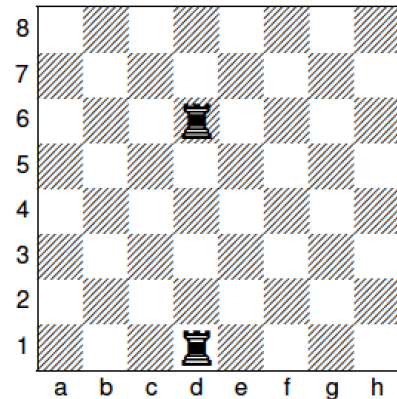
$$(8 \cdot 7 \cdot 6)^2$$

# Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don't share a row or a column



(a) valid



(b) invalid

Poll:

A.  $8^2 \cdot 7^2$

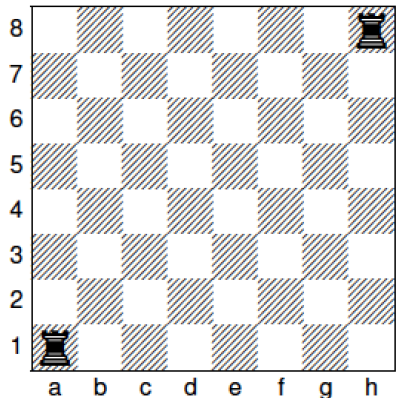
B.  $\binom{8}{2} \cdot \binom{8}{2}$

C.  $\frac{8^2 \cdot 7^2}{2}$

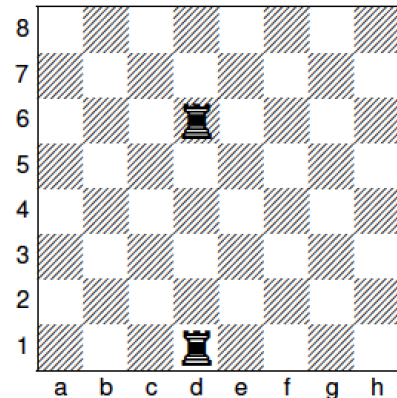
D. I don't know.

## Rooks on chessboard - solution

How many ways to place two identical rooks on a chessboard so that they don't share a row or a column



(a) valid



(b) invalid

**Pretend Rooks are different**

1. Column for rook1
2. Row for rook1
3. Column for rook2
4. Row for rook2

$$(8 \cdot 7)^2$$

**Remove the order between two rooks**

$$(8 \cdot 7)^2 / 2$$

## Example – Sum of integers

*“How many solutions  $(x_1, \dots, x_k)$  such that  $x_1, \dots, x_k \geq 0$  and  $\sum_{i=1}^k x_i = n$ ?”*

**Example:**  $k = 3, n = 5$

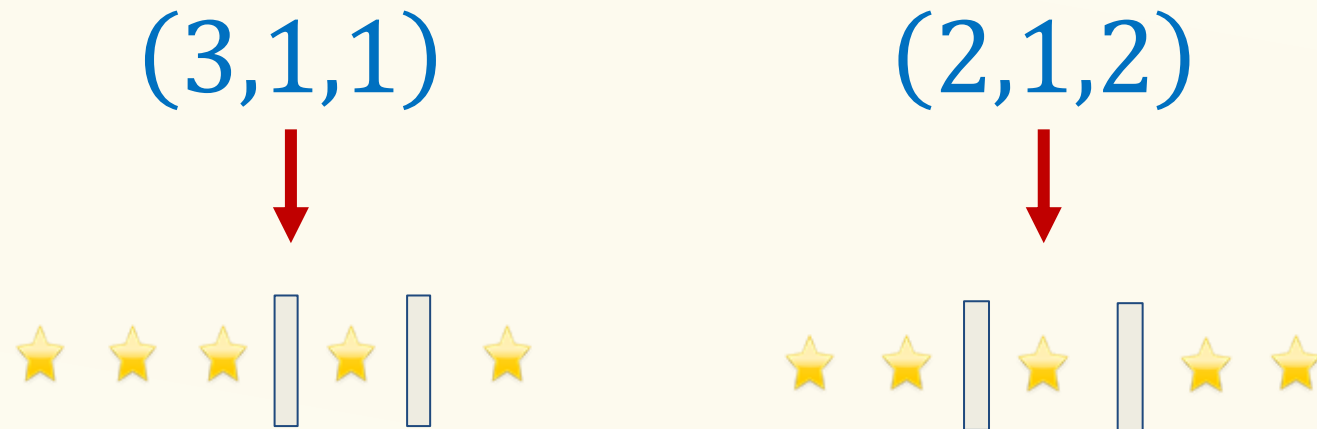
$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \dots$

## Example – Sum of integers (k=3, n=5) - solution

**Example:**  $k = 3, n = 5$

# sols = # strings from  $\{0,1\}^7$  w/ exactly two 0s =  $\binom{7}{2} = 21$

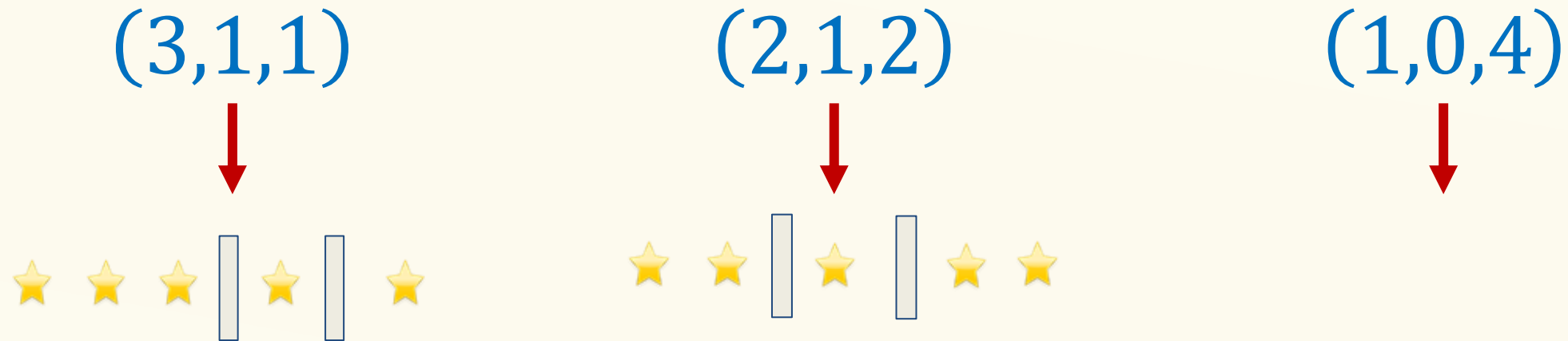
**Clever representation of solutions with stars and bars**



## Example – Sum of integers ( $k=3, n=5$ )

**Example:**  $k = 3, n = 5$

**Clever representation of solutions (with stars and bars)**



## Example – Sum of integers – general case

“How many solutions  $(x_1, \dots, x_k)$  such that  $x_1, \dots, x_k \geq 0$  and  $\sum_{i=1}^k x_i = n$ ?”

$$\begin{aligned} \# \text{ sols} &= \# \text{ strings from } \{0,1\}^{n+k-1} \text{ w/ } k-1 \text{ os} \\ &= \binom{n+k-1}{k-1} \end{aligned}$$

After a change in representation, the problem magically reduces to counting combinations.

## Challenge problem

Use the stars and bars approach to count the number of ways to choose  $k$  out of  $n$  distinct objects if repetition is allowed (but don't care about order)

# Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Sleuth's criterion
- Stars and bars
- Pigeonhole principle

# Counting is NOT for kindergarteners

