More Counting!

CSE 312 Spring 26

Lecture 3

We will use
PollEv.com/annakarlin185
Today again

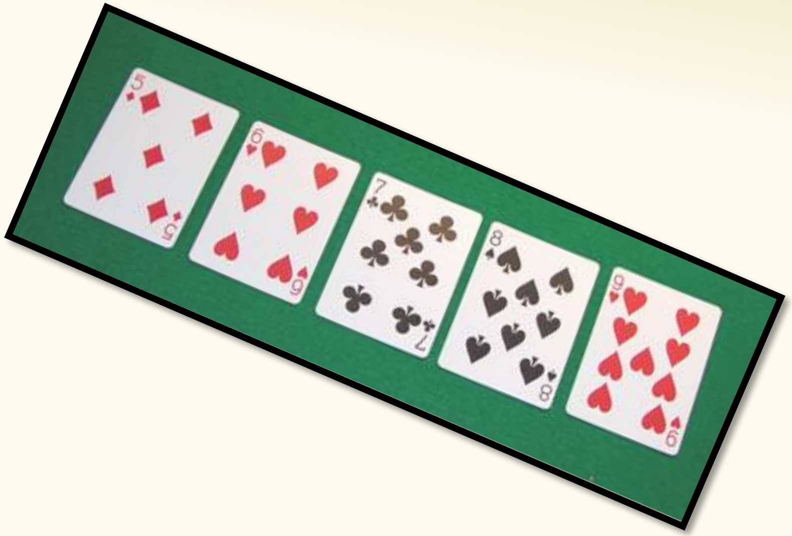
Recap

- Sum Rule, Product Rule
- Combinations
- Binomial Theorem
- Multinomial Coefficients
- Combinatorial Proofs
- Inclusion-Exclusion

Today's Agenda (1)

- Sleuth's Criterion
- Stars and Bars
- Pigeonhole Principle
- More examples?

Quick Review of Cards



How many possible 5 card hands?

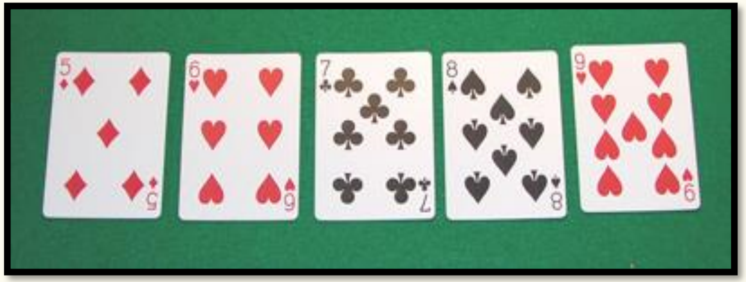
$\binom{52}{5}$

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

Counting Cards I

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A **straight** is five consecutive rank cards of any suit. How many possible straights?



choose lowest rank 10

choose suit for lowest 4

" 2nd lowest

suit 5th 4

$10 \cdot 4^5$

Counting Cards II

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

A **flush** is a five card hand all of the same suit.

How many possible flushes?

pick suit 4

pick 5 ranks $\binom{13}{5}$

$4 \cdot \binom{13}{5}$

Counting Cards III

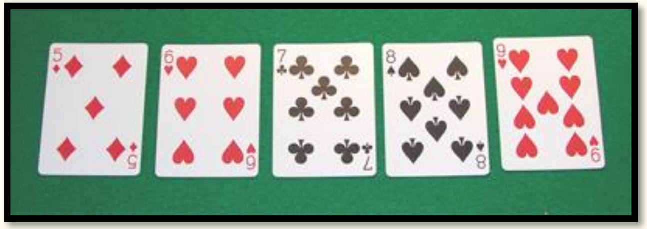
- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

A **flush** is five card hand all of the same suit.

How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

How many flushes are NOT straights?



flushes − # straight flushes

$4 \cdot \binom{13}{5}$ $\qquad$ $10 \cdot 4$

Counting Cards IV

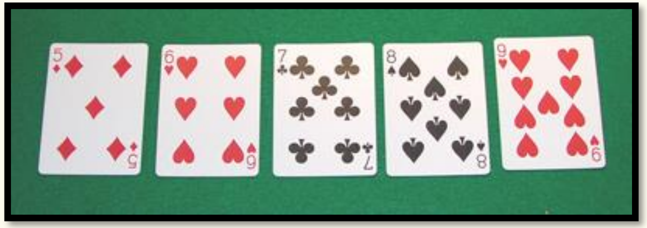
- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A flush is five card hand all of the same suit. How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

- How many flushes are NOT straights?

= #flush - #flush and straight



$$\left(4 \cdot \binom{13}{5} = 5148\right) - 10 \cdot 4$$

Sleuth's Criterion (Rudich) - example

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

$$\binom{4}{3} \cdot \binom{49}{2}$$

Sleuth's Criterion (Rudich) – poll on example

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

$$\binom{4}{3} \cdot \binom{49}{2}$$

1. First choose 3 Aces.
2. Then choose remaining two cards.

Poll:

A. Correct
B. Overcount
C. Undercount

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The answer

1. First choose 3 Aces.

2. Then choose remaining two cards.

$$\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 48$$

AH AD AS 2C 10H

AH AD AS AC 2C

overcounted hands

48

each counted 4 x

Sleuth's Criterion (Rudich) - a different way

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence → under counting

Many sequences → over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

When in doubt, break up into disjoint sets you know how to count, and then use the sum rule.

Sleuth's Criterion (Rudich) – correct solution

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

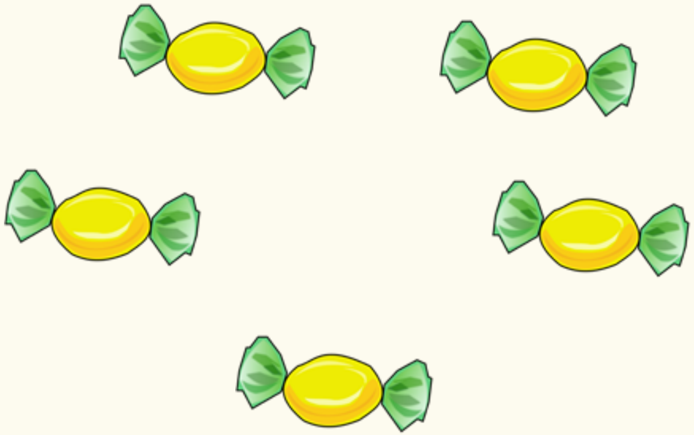
= # 5 card hand containing exactly 3 Aces $\binom{4}{3} \cdot \binom{48}{2}$

+ # 5 card hand containing exactly 4 Aces $\binom{48}{1}$

Agenda (1)

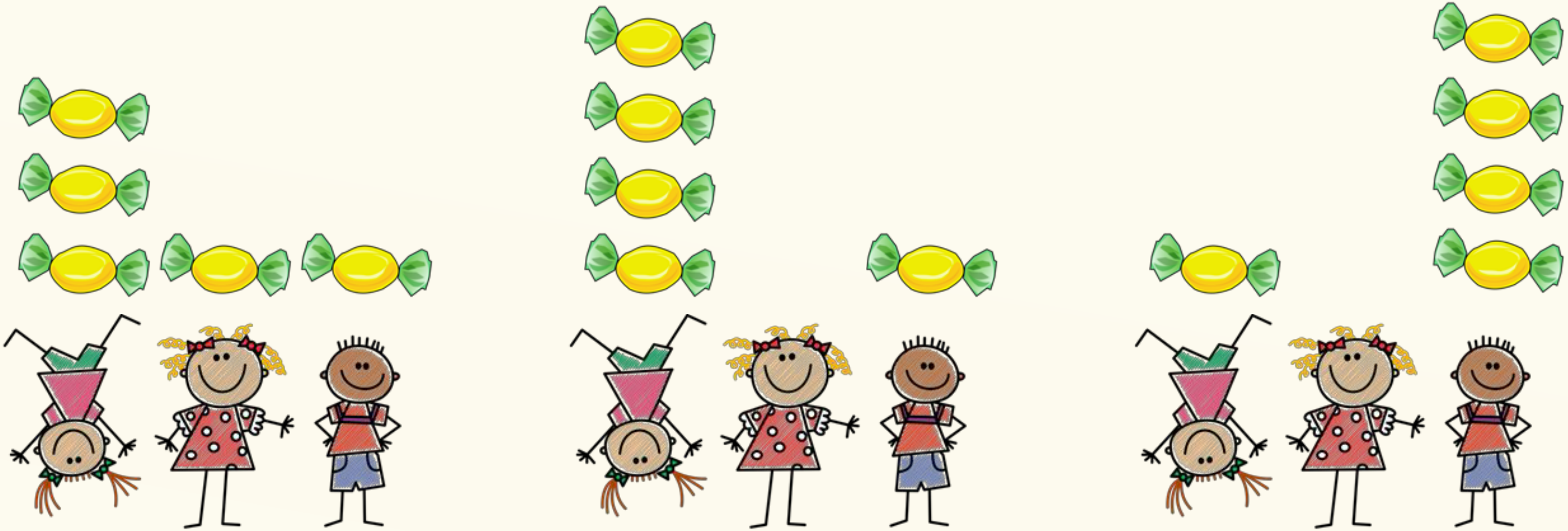
- Sleuth's Criterion
- Stars and Bars
- Pigeonhole Principle
- More examples?

Example: Kids and Candies

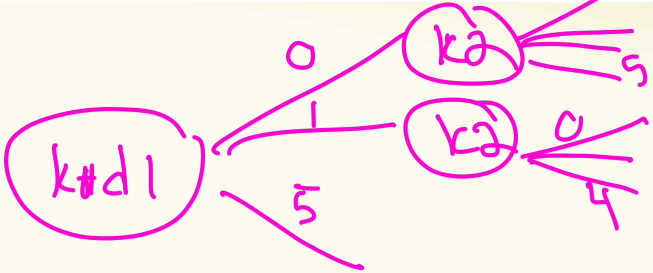


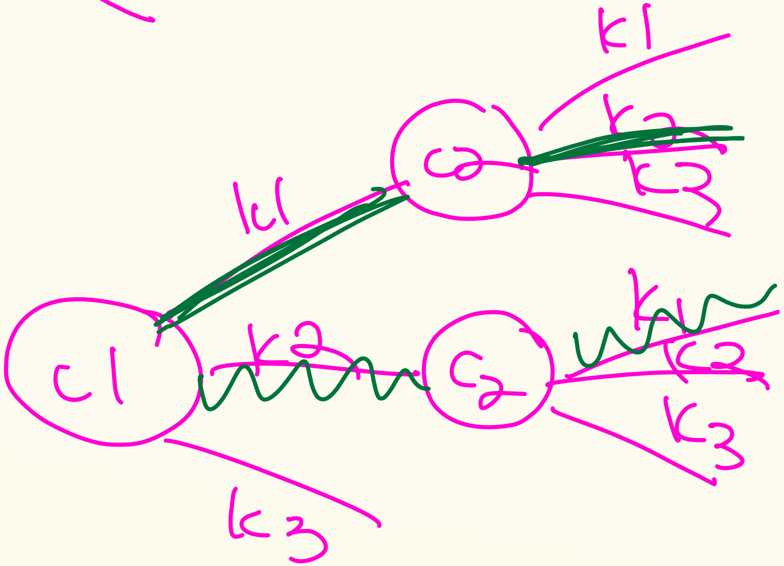
How many ways can we give five **indistinguishable** candies to these three kids?

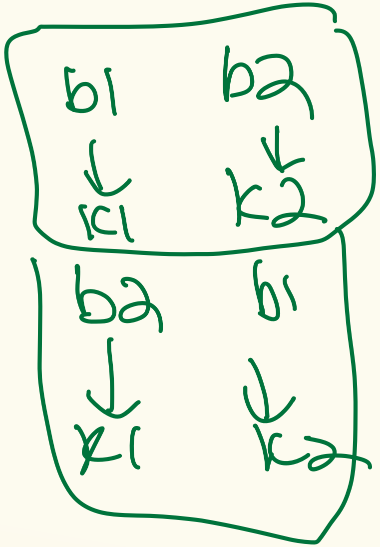
Kids + Candies

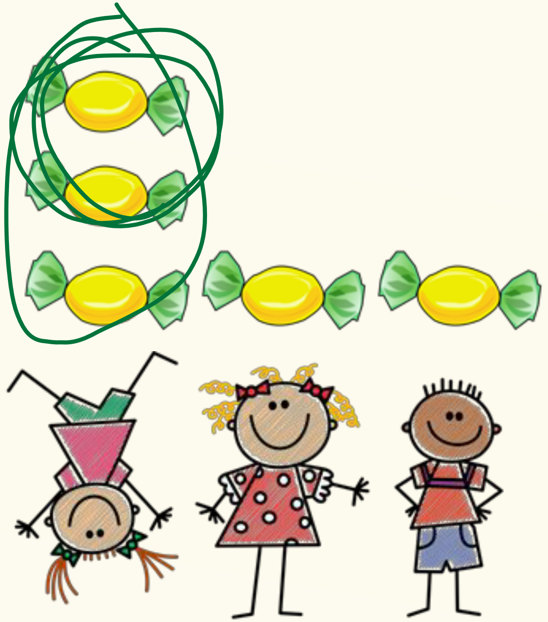


Kids + Candies





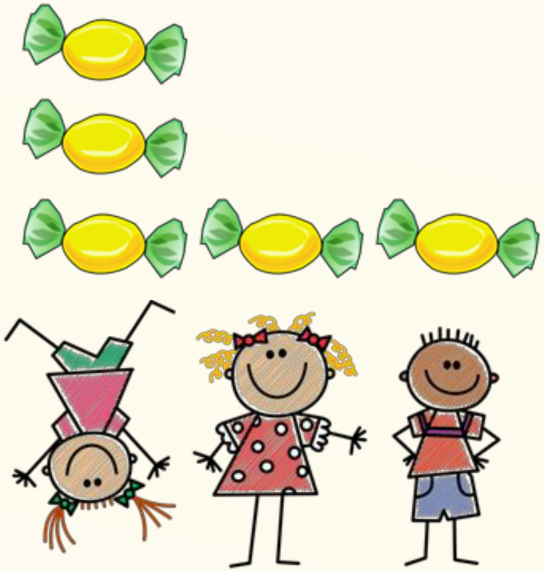




$$\frac{3^5}{5!}$$

Kids + Candies

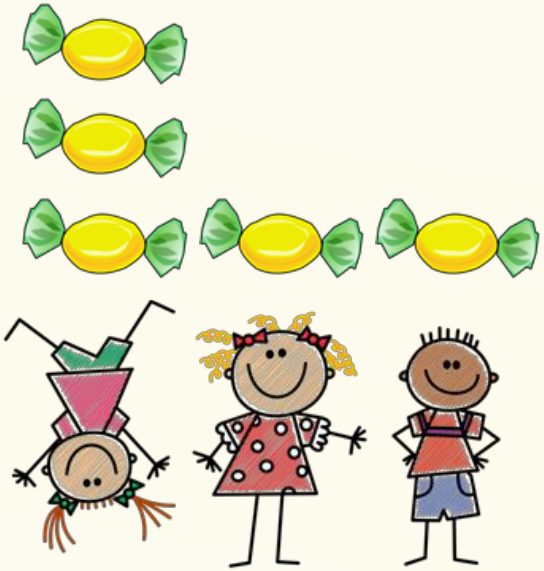
- Idea: count something different



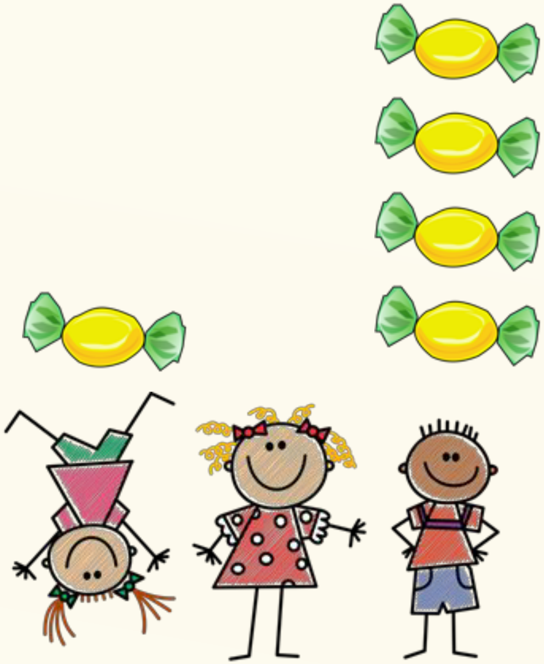
Kids + Candies: count something equivalent

Idea: Count something equivalent

5 “stars” for candies, 2 “bars” for dividers.



Kids + Candies: another example

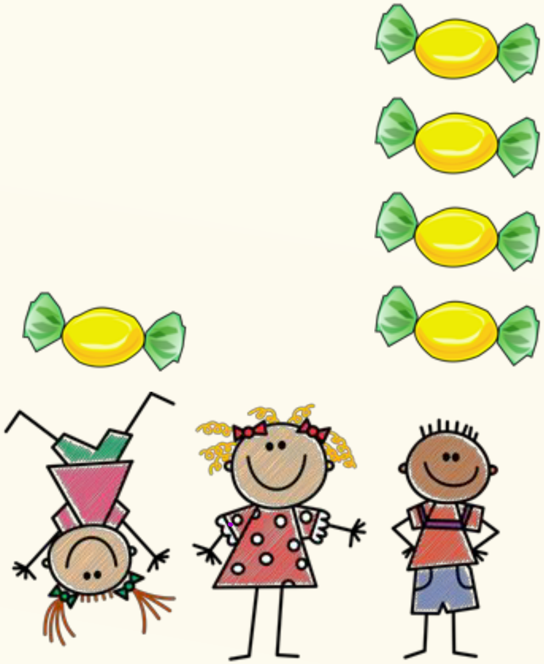


Idea: Count something equivalent

5 “stars” for candies, 2 “bars” for dividers.

Kids + Candies: bijection

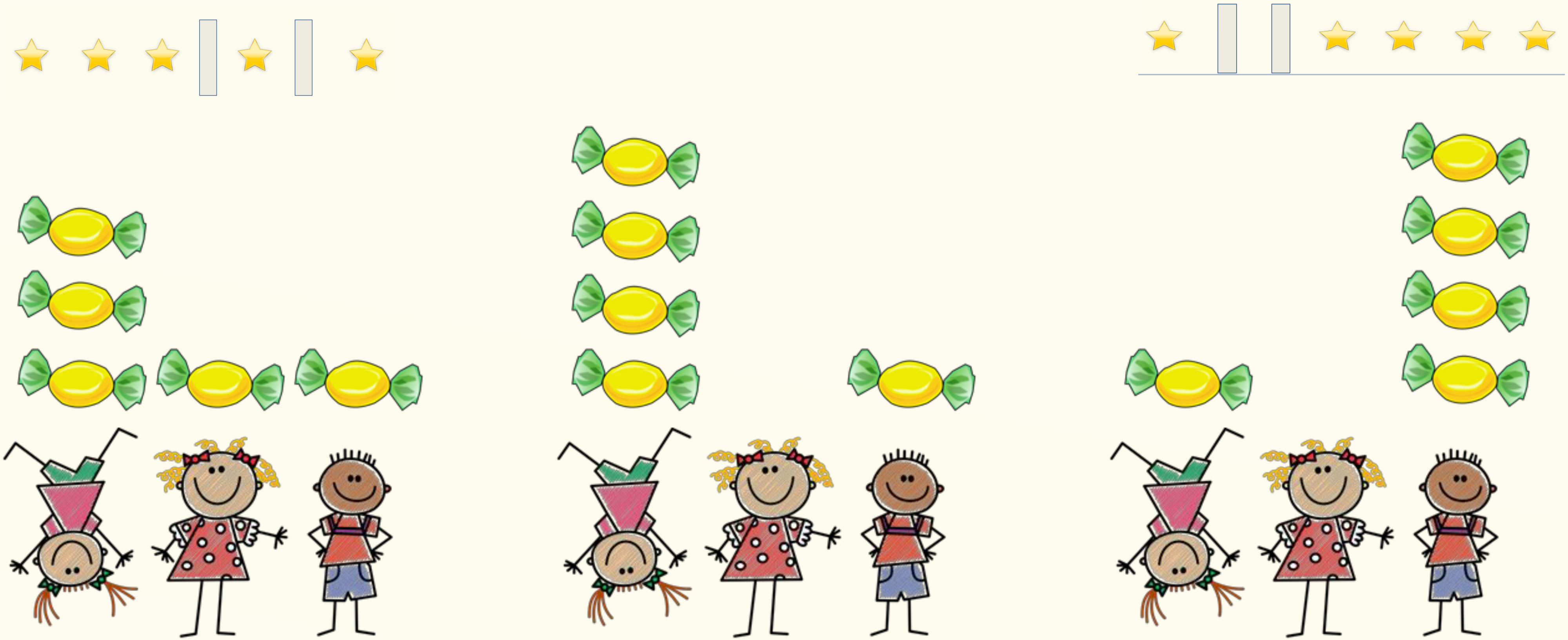
$$\binom{7}{2} = \binom{7}{5}$$



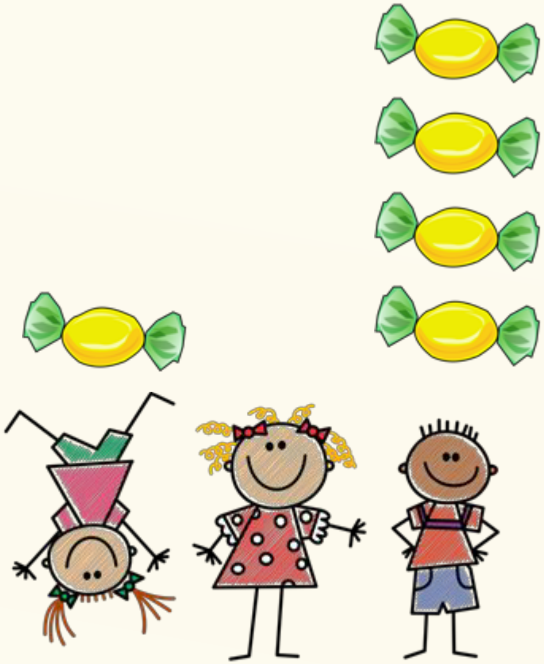
For each candy distribution, there is exactly one corresponding way to arrange the stars and bars.

Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.

Kids + Candies



Kids + Candies: conclusion



Hence, the number of ways to distribute candies to the 3 kids is the number of arrangements of stars and bars.

This is

$$\binom{7}{2} = \binom{7}{5}$$

Stars and Bars / Divider method

candies

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

↑ kids

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Agenda (3)

- Sleuth’s Criterion
- Stars and Bars
- Pigeonhole Principle
- More examples?

Pigeonhole Principle (PHP): Idea

10 pigeons into 9 pigeonholes

There must be at least one pigeonhole with at least two pigeons

Pigeonhole Principle

If there are n pigeons in $n - 1$ holes or fewer, then one hole must contain at least 2 pigeons!

Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

Pigeonhole Principle – example repeated

In a room with 367 people, there are at least two with the same birthday.

Solution:

1. **367** pigeons = people
2. **366** holes = possible birthdays (including leap day)
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

If there are n pigeons in $n - 1$ holes or fewer, then one hole must contain at least 2 pigeons!

Pigeonhole Principle – More generally

If there are n pigeons in k < n holes, then one hole must contain at least n/k pigeons!

Proof. Assume there are < n/k pigeons per hole.

Then, there are < k(n/k) = n pigeons overall.

Contradiction!

Pigeonhole Principle – Better version

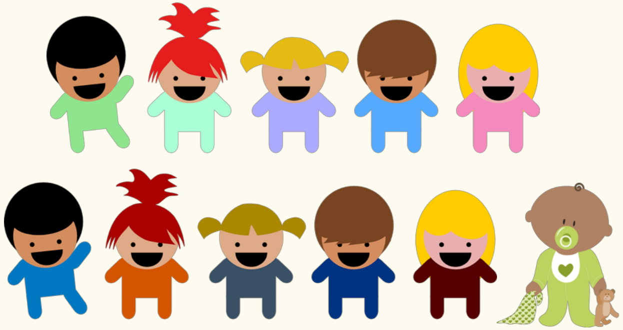
If there are n pigeons in k < n holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

Reason. Can’t have fractional number of pigeons

Syntax reminder:

- Ceiling: $\lceil x \rceil$ is x rounded up to the nearest integer (e.g., $\lceil 2.731 \rceil = 3$)
- Floor: $\lfloor x \rfloor$ is x rounded down to the nearest integer (e.g., $\lfloor 2.731 \rfloor = 2$)

Pigeonhole Principle: Example 2



How many children must share a cake in the worst case?

1. Pigeons: 11 children   n
2. Pigeonholes: k=3 cakes.   k
3. Kid goes to cake they will be sharing.
4. By PHP, there will be one cake shared by at least $\lceil 11/3 \rceil = 4$ kids.

$3\frac{2}{3}$

Pigeonhole Principle – Example 3

In every set S of 100 integers, there are at least ***two*** *elements whose difference is a multiple of 37.*

$S = \{5, 2^{37}, -101, 23, \ldots\}$

$a \equiv 6 \mod 37$ $\qquad$ $b \equiv 6 \mod 37$

$a - b \equiv 6 - 6 \mod 37 \equiv 0 \mod 37$

Pigeonhole Principle – Example 3 - setup

In every set S of 100 integers, there are at least ***two*** *elements whose difference is a multiple of 37.*

When solving a PHP problem:

1. Identify pigeons — elts of S
2. Identify pigeonholes — $i \bmod 37$, $i = 0, 1, \ldots, 36$
3. Specify how pigeons are assigned to pigeonholes — $x \in S \to x \bmod 37$
4. Apply PHP — $n = 100$, $k = 37$, $\left\lceil \frac{100}{37} \right\rceil = 3$

Pigeonhole Principle – Example 3 (Surprising?)

*In every set S of 100 integers, there are at least **two** elements whose difference is a multiple of 37.*

When solving a PHP problem:

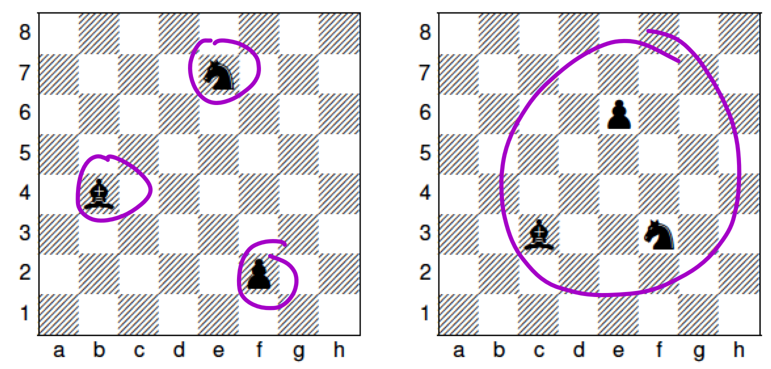
1. Identify pigeons	integers in set S
2. Identify pigeonholes	value mod 37
3. Specify how pigeons are assigned to pigeonholes	x to hole x mod 37
4. Apply PHP	100 pigeons to 37 PHs

Agenda (4)

- Sleuth’s Criterion
- Stars and Bars
- Pigeonhole Principle
- More examples?

8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column ?



(a) valid (b) invalid

Poll:

A. $\binom{64}{3}$

B. $\binom{8}{3} \cdot \binom{8}{3}$

C. $8^2 \cdot 7^2 \cdot 6^2$

D. I don't know.

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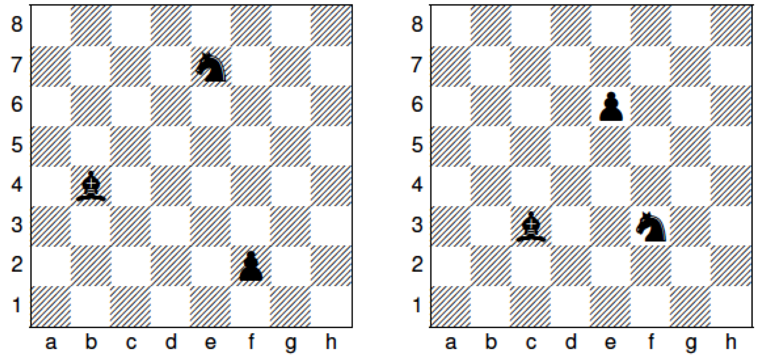
$\binom{64}{3}$ choose 3 pos

$\binom{8}{3}\binom{8}{3}$ choose 3 rows
choose 3 cols

8 by 8 chessboard - solution

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column ?



(a) valid (b) invalid

Sequential process:

1. Column for pawn
2. Row for pawn
3. Column for bishop
4. Row for bishop
5. Column for knight
6. Row for knight

$(8 \cdot 7 \cdot 6)^2$

Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Sleuth’s criterion
- Stars and bars
- Pigeonhole principle

Counting is NOT for kindergarteners