Student name: ________________ **Student Number:** ________________

CSE 312: Foundations of Computing II — Spring 2026

Practice Problems for final

Important: Read the instructions on this page carefully, but do not turn the page until you are instructed to do so.

Instructions. You have 1 hour and 50 minutes to complete this final (though the exam has been designed for a 70 minute slot.)

- **Write your name and student number on top of this page.**
- **If you do not have an ID with a picture, you will not be able to take the exam.**
- This is a **closed-book, closed notes exam** with the exception of the cheat sheet that we are giving you.
- **No electronics are allowed** during the exam (no smart phones, no laptops, no smart watches, no pocket calculators, etc). **Before you come into the room** where you are taking the test, store them in your bag/backpack and do not take them out until you leave the room after the exam. If we see any such items once you are in the room where you take the test, we will take your test away.
- Write your final solutions in the appropriate boxes. Be sure that what you put in the box is **only** your final answer and is written neatly.
- If your final answer to a question is correct, you will get full credit *regardless of whether or not you provide any explanation*. On some of the problems (**but not all**), we will provide a little bit of partial credit if you provide correct partial explanations. There will be no partial credit on any True/False or multiple choice problem.
- If a problem looks difficult, I recommend moving on to another problem and coming back later.

Good luck!

Task 1 – Law of total expectation

Sinho and Vrettos are playing a game where they each choose an integer uniformly at random from $[0, 100]$, then whoever has the larger number wins (and it's a draw if they tie). However, Vrettos doesn't like losing, so he's rigged his random number generator such that it instead picks uniformly at random from the integers that are between Sinho's number and 100, inclusive. Let S be Sinho's number and V be Vretto's number.

1. What is $\mathbb{E}[S]$?
2. What is $\mathbb{E}[V|S = s]$, where s is any integer such that $0 \leqslant s \leqslant 100$?
3. What is $\mathbb{E}[V]$? (You can leave your answer as a sum.)
4. What is the probability that the game ends in a draw?(You can leave your answer as a sum)

1. S is a discrete uniform random variable between 0 and 100, so its expectation is $\frac{0+100}{2} = 50$.
2. If $S = s$, we know that V will be uniformly distributed between s and 100. $(E(V|S = s) = \sum_{v=s}^{v=100} vPr(v = v|S = s)$. Similar to the previous part, this gives us that $\mathbb{E}[V|S = s] = \frac{s+100}{2}$.
3. Using the law of total expectation, we can say:

$$\mathbb{E}[V] = \sum_{s=0}^{100} \mathbb{E}[V|S = s] \cdot \mathbb{P}(S = s)$$

$$\mathbb{E}[V] = \sum_{s=0}^{100} \frac{s + 100}{2} \cdot \frac{1}{101}$$

$$\mathbb{E}[V] = \frac{1}{202} \left(\sum_{s=0}^{100} s + \sum_{s=0}^{100} 100 \right)$$

   The first summation comes out to $\frac{100(100+1)}{2}$. The second summation is just adding 100 to itself 101 times, so it comes out to $100 \cdot 101$. Plugging these values in, we get $\mathbb{E}[V] = 75$.

4.

$$\mathbb{P}(\text{draw}) = \sum_{s=0}^{100} \mathbb{P}(\text{draw}|S = s) \mathbb{P}(S = s) = \sum_{s=0}^{100} \mathbb{P}(V = s|S = s) \mathbb{P}(S = s) \quad (1)$$

$$= \sum_{s=0}^{100} \frac{1}{(100 - s + 1)} \cdot \frac{1}{101} \quad (2)$$

Task 2 – No partial credit on this task...

Write any calculations for this problem on scratch paper. **Circle one of True or False.**

1. **True or False:** Suppose that $f(x)$ is a function s.t. $\int_{-\infty}^{\infty} f(x)dx = 1$. Then $f(x)$ is a valid probability density function for some random variable X.

   False $f(x)$ needs to be nonnegative.

2. **True or False:** Let $X_1, \ldots, X_n$ be i.i.d. from some distribution, each with density $f_X(x)$ and CDF $F_X(x)$. Let $Y = \max(X_1, \ldots, X_n)$. Then the CDF of Y is

$$F_Y(y) = (F_X(y))^n.$$

   True.

$$\mathbb{P}(Y \leqslant y) = \mathbb{P}(\max(X_1, \ldots, X_n) \leqslant y) = \mathbb{P}(X_1 \leqslant y, \ldots, X_n \leqslant y)$$

   which by independence is

$$\prod_{i=1}^{n} \mathbb{P}(X_i \leqslant y) = (F_X(y))^n$$

3. **True or False:** Let A and B be events such that $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.6$ and $\mathbb{P}(A \cup B) = 0.8$. Then the events A and B are independent.

   True. We can use inclusion-exclusion to compute

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.5 + 0.6 - 0.8 = 0.3\,.$$

   At the same time, note that $\mathbb{P}(A) \cdot \mathbb{P}(B) = 0.5 \times 0.6 = 0.3$, and thus A, B are independent.

4. **True or False:** If X and Y are independent random variables, each taking values in $\{+1, -1\}$, then

$$\mathbb{E}\left[(X-Y)^2\right] = 2 - 2\,\mathbb{E}[X] \cdot \mathbb{E}[Y]\,.$$

   True. First, note that by linearity of expectation

$$\mathbb{E}\left[(X-Y)^2\right] = \mathbb{E}\left[X^2 - 2XY + Y^2\right] = \mathbb{E}\left[X^2\right] - 2\,\mathbb{E}[XY] + \mathbb{E}\left[Y^2\right]\,.$$

   Because X, Y are independent, we also have $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$. Because each of X and Y is 1 or -1, $X^2 = Y^2 = 1$.

5. **True or False:** For any random variable X, $E(5^X) = 5^{E(X)}$.

   False: The only functions for which $E(g(X)) = g(E(X))$ for all random variables X are linear functions.

6. **True or False:** Let X be a normal random variable with parameters μ (mean) and σ^2 (variance). Then,

$$\mathbb{P}\left(|X-\mu| \geqslant k\sigma\right) = 1 - \Phi(k) + \Phi(-k)\ .$$

   True: For a normal random variable $\mathcal{N}(\mu, \sigma^2)$, $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. Then

$$\begin{aligned}\mathbb{P}\left(|X-\mu| \geqslant k\sigma\right) &= \mathbb{P}\left(|Z| \geqslant k\right) = \mathbb{P}\left(Z \leqslant -k\right) + \mathbb{P}\left(Z \geqslant k\right) \\ &= \Phi(-k) + 1 - \mathbb{P}\left(Z \leqslant k\right) = \Phi(-k) + 1 - \Phi(k)\end{aligned}$$

7. Multiple Choice: Suppose that $X_1, \ldots, X_n$ are i.i.d. samples from a normal distribution with unknown mean μ and variance 9. How big does n need to be so that μ is in $[\overline{X} - 0.03, \overline{X} + 0.03]$ with probability at least 0.97. (As usual $\overline{X} = (\sum_{i=1}^{n} X_i)/n$.) You should use the fact that $\Phi^{-1}(0.985) = 2.17$. **Circle the correct answer.**

   (a) $n \geqslant \frac{9}{0.03} \cdot 2.17$.

   (b) $n \geqslant \left(\frac{3}{0.03} \cdot 2.17\right)^2$.

   (c) $n \geqslant \left(\frac{0.03}{3} \cdot 2.17\right)^2$.

   (d) $n \geqslant \left(\frac{0.03}{3} \cdot \frac{1}{2.17}\right)^2$.

   The answer is (b).
   $\overline{X}$ has mean μ and variance $9/n$. So we want,

$$Pr(\mu - 0.03 \leqslant \overline{X} \leqslant \mu + 0.03) = Pr\left(-\frac{0.03}{\sqrt{\frac{9}{n}}} \leqslant Z \leqslant \frac{0.03}{\sqrt{\frac{9}{n}}}\right) \geqslant 0.97.$$

   Let

$$a = \frac{0.03}{\sqrt{\frac{9}{n}}}.$$

   Then we want

$$\Phi(a) - \Phi(-a) = \Phi(a) - (1 - \Phi(a)) = 2\Phi(a) - 1 \geqslant 0.97$$

. Or equivalently

$$\Phi(a) \geqslant 1.97/2 = 0.985$$

or equivalently

$$a \geqslant \Phi(0.985) = 2.17.$$

Plugging back in for a, we have

$$\frac{0.03}{\sqrt{\frac{9}{n}}} \geqslant 2.17 \quad \equiv \quad \sqrt{n} \geqslant \frac{3}{0.03} \cdot 2.17 \quad \equiv \quad n \geqslant \left(\frac{3}{0.03} \cdot 2.17\right)^2$$

8. **True or False:** Suppose that X and Y are independent continuous random variables where X has CDF $F_X(x)$ and Y has pdf $f_Y(y)$.

$$Pr(X > 3Y) = \int_{-\infty}^{\infty} [1 - F_X(3y)] f_Y(y) dy.$$

True: The law of total probability (continuous) says

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A|X = x) f_X(x) dx$$

Apply this to the event $X > 3Y$, conditioning on $Y = y$. Then

$$Pr(X > 3Y) = \int_{-\infty}^{\infty} Pr(X > 3Y|Y = y) f_Y(y) dy = \int_{-\infty}^{\infty} Pr(X > 3y|Y = y) f_Y(y) dy$$

which by independence

$$= \int_{-\infty}^{\infty} Pr(X > 3y) f_Y(y) dy = \int_{-\infty}^{\infty} [1 - F_X(3y)] f_Y(y) dy$$

9. **True or False:** Suppose that X is a continuous random variable with pdf $f_X(x)$. Then $0 \leqslant f_X(x) \leqslant 1$ for all real x.

False: $f(x)$ can be greater than 1.

10. **True or False:** For any continuous probability distribution on the reals $Pr(X = c) = 0$ for all real values of c.

True

11. **True or False:** If Z is normally distributed with mean 1 and variance 4 and a is some real number, then

$$Pr(Z < -a) = Pr(Z > a).$$

False: A normal random variable has a density that is symmetric around its mean, not around 0 (unless it has mean 0).

12. **True or False:** Let $f_{X,Y}(x, y)$ be the joint density function of two random variables X and Y. Then

$$F_X(a) = Pr(X \leqslant a) = \int_{-\infty}^{a} f_{X,Y}(x, y)\ dy.$$

False. Should be

$$F_X(a) = \int_{-\infty}^{a} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\ dy\ dx.$$

13. **True or False:** Suppose that X, Y, Z are continuous random variables with joint density $f_{X,Y,Z}(x, y, z)$. Then the marginal density function of X is

$$f_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y,Z}(x, y, z)\ dy\ dz.$$

True:

Task 3 – MLE

Let $X_1 = x_1, \ldots, X_n = x_n$ be i.i.d. samples from a random variable that follow a so-called Borel distribution with unknown parameter θ, i.e., a distribution with probability mass function

$$\mathbb{P}\left(X = x; \theta\right) = \frac{e^{-\theta x}(\theta x)^{x-1}}{x!} \,,$$

where $0 < \theta \leqslant 1$ is a real number, and $x \geqslant 1$ is an integer. (Note that the values Borel random variables take are positive integers.) What is the maximum likelihood estimator for θ?

The likelihood is

$$\mathcal{L}(x_1, x_2, \ldots, x_n\,; \theta) = \prod_{i=1}^{n} \frac{e^{-\theta x_i}(\theta x_i)^{x_i-1}}{x_i!} \,.$$

Therefore the log-likelihood is

$$\ln \mathcal{L}(x_1, x_2, \ldots, x_n\,; \theta) = \sum_{i=1}^{n} \left[-\theta x_i + (x_i - 1)(\ln(\theta) + \ln(x_i)) - \ln(x_i!)\right]$$

We take the derivative of the log-likelihood with respect to the parameter θ:

$$\frac{\partial}{\partial \theta} \ln \mathcal{L}(x_1, x_2, \ldots, x_n\,; \theta) = \sum_{i=1}^{n} \left(-x_i + \frac{x_i - 1}{\theta}\right) \,.$$

Now, we set the derivative to 0 and solve (here we replace θ with $\hat{\theta}$:

$$\sum_{i=1}^{n} \left(-x_i + \frac{x_i - 1}{\theta}\right) = 0$$

$$\frac{1}{\hat{\theta}} \sum_{i=1}^{n} (x_i - 1) = \sum_{i=1}^{n} x_i$$

Therefore

$$\boxed{\hat{\theta} = \frac{\sum_{i=1}^{n}(x_i - 1)}{\sum_{i=1}^{n} x_i} = 1 - \frac{n}{\sum_{i=1}^{n} x_i}} \,.$$

Check that it's a global maximum (not required):

$$\frac{\partial^2}{\partial \theta^2} \ln \mathcal{L}(x_1, x_2, \ldots, x_n\,; \theta) = -\sum_{i=1}^{n} \frac{x_i - 1}{\theta^2} < 0$$

so $\ln \mathcal{L}(x_1, x_2, \ldots, x_n\,; \theta)$ is concave downward everywhere.

Task 4 – Random Frog

A frog takes a random walk on an infinite (discrete) line, where coordinates correspond to all integers (positive and negative), i.e., the elements of $\mathbb{Z}$, with the numbers ordered as usual increasing to the right and decreasing to the left.

- The frog starts at the coordinate $x = 0$.
- At every step, the frog jumps 2 to the right with probability $2/3$, and jumps 1 to the left with probability $1/3$. That is, the frog jumps from x to x' where $x' - x$ is 2 with probability $2/3$ and -1 with probability $1/3$.

Let X be the position of the frog after 200 steps.(Note that each jump is independent of all other jumps.)

(NOTE: One could model the frog's position as a Markov chain, but you should NOT use the language of Markov chains here! Remember to briefly justify your answers.)

1. Use linearity of expectation to compute $\mathbb{E}[X]$.

   We can write $X = \sum_{i=1}^{200} X_i$, where $X_i \in \{-1, 2\}$, and $\mathbb{P}(X_i = -1) = 1/3$ and $\mathbb{P}(X_i = 2) = 2/3$. Then, $\mathbb{E}[X_i] = -1/3 + 4/3 = 1$. By linearity, $\mathbb{E}[X] = 200 \cdot 1 = 200$.

2. What is $\text{Var}(X)$?

   We use the same notation as in the solution of **a)**. Because the X_i's are independent, $\text{Var}(X) = 200 \cdot \text{Var}(X_i)$. Further, by LOTUS, $\mathbb{E}\left[X_i^2\right] = 1/3 \cdot 1 + 2/3 \cdot 4 = 1/3 + 8/3 = 3$. And therefore, $\text{Var}(X_i) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = 3 - 1^2 = 2$. Therefore, $\text{Var}(X) = 200 \cdot 2 = 400$.

3. Use the CLT, including the continuity correction, to estimate the probability that after 200 steps the frog is at a position that is 250 or larger, i.e., to estimate $\mathbb{P}(X \geqslant 250)$. Write your result as a function of Φ. (You do NOT need to simplify numerical values.)

   Since X has mean 200 and standard deviation $\sigma = 20$, we approximate X with a random variable $Z \sim \mathcal{N}(200, 20^2)$, and the desired probability with $\mathbb{P}(Z \geqslant 249.5)$. Then, we compute

$$\mathbb{P}(Z \geqslant 249.5) = \mathbb{P}(Z - 200 \geqslant 49.5) = \mathbb{P}\left(\frac{Z - 200}{20} \geqslant \frac{49.5}{20}\right) = 1 - \Phi(\frac{49.5}{20}) = 1 - \Phi(2.475)\,.$$

Task 5 –

1. Suppose that 100 distinct balls are thrown independently and uniformly at random into 100 distinct bins. What is the probability that bin 1 has 5 balls in it given that bin 2 has 3 balls in it? (Use the following notation: Let B_i be the number of balls in bin i.)

$$\mathbb{P}(B_1 = 5 | B_2 = 3) = \frac{\binom{100}{5}\binom{95}{3}\left(\frac{1}{100}\right)^8\left(\frac{98}{100}\right)^{92}}{\binom{100}{3}\left(\frac{1}{100}\right)^3\left(\frac{99}{100}\right)^{97}}.$$

This expression can be simplified quite easily.

2. Every minute, a random word generator spits out one word uniformly at random from the 3-word set { I , love, to}. The word spit out is independent of words spit out at other times. Let X be the number of times that the phrase "I love to love" appears if we let the generator run for n minutes. What is $\mathbb{E}[X]$?

Let X_i be an indicator random variable that is 1 if the words $i, i+1, i+2, i+3$ are "I love to love". Then $E(X_i) = (1/3)^4$ and $X = \sum_{i=1}^{n-3} X_i$. Thus,

$$E(X) = \sum_i E(X_i) = (n-3)\frac{1}{3^4}.$$