

Application: Distinct Elements

CSE 312 Spring 26

Lecture 21

Data mining – Stream Model

- In many data mining situations, data often not known ahead of time.
 - Examples: Google queries, Twitter or Facebook status updates, YouTube video views
- Think of the data as an infinite stream
- Input elements (e.g. Google queries) enter/arrive one at a time.
 - We cannot possibly store the stream.

Question: How do we make critical calculations about the data stream using a limited amount of memory?

Stream Model – Problem Setup

Input: sequence (aka. “stream”) of N elements x_1, x_2, \dots, x_N from a known universe U (e.g., 8-byte integers).

Goal: perform a computation on the input, in a single left to right pass, where:

- Elements processed in real time
- Can’t store the full data \Rightarrow use minimal amount of storage while maintaining working “summary”

What can we compute?

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

Some functions are easy:

- Min
- Max
- Sum
- Average

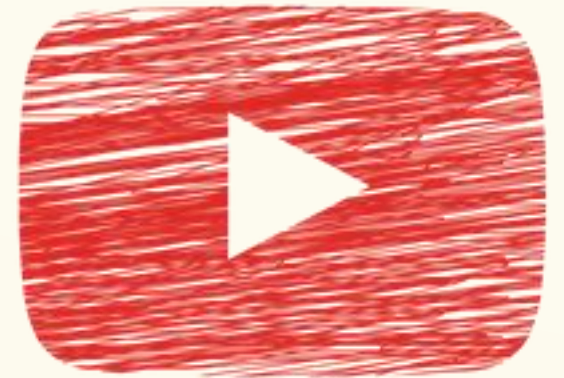
Today: Counting distinct elements

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

Application

You are the content manager at YouTube, and you are trying to figure out the **distinct** view count for a video. How do we do that?

Note: A person can view their favorite videos several times, but they only count as 1 **distinct** view!



Other applications

- IP packet streams: How many distinct IP addresses or IP flows (source+destination IP, port, protocol)
 - Anomaly detection, traffic monitoring
- Search: How many distinct search queries on Google on a certain topic yesterday (users identified by IP address)
- Web services: how many distinct users (cookies) searched/browsed a certain term/item
 - Advertising, marketing trends, etc.

Counting distinct elements

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

N = # of IDs in the stream = 11, m = # of distinct IDs in the stream = 5

Want to compute number of **distinct** IDs in the stream.

How?

Counting distinct elements

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

N = # of IDs in the stream = 11, m = # of distinct IDs in the stream = 5

Want to compute number of **distinct** IDs in the stream.

- Naïve solution: As the data stream comes in, store all distinct IDs in a hash table.
- Space requirement: $\Omega(m)$

YouTube Scenario: m is huge!

Counting distinct elements

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

N = # of IDs in the stream = 11, m = # of distinct IDs in the stream = 5

Want to compute (at least approximately) the number of **distinct** IDs in the stream.

How to do this without storing all the elements?

Detour – I.I.D. Uniforms

If $Y_1, \dots, Y_m \sim \text{Unif}(0,1)$ (i.i.d.) where do we expect the points to end up?

$m = 1$



Detour – I.I.D. Uniforms

If $Y_1, \dots, Y_m \sim \text{Unif}(0,1)$ (i.i.d.) where do we expect the points to end up?

$m = 1$



$m = 2$

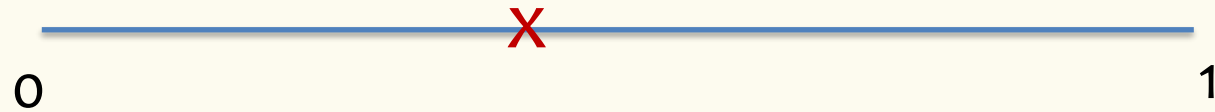


Detour – I.I.D. Uniforms

If $Y_1, \dots, Y_m \sim \text{Unif}(0,1)$ (i.i.d.) where do we expect the points to end up?

“Evenly spread out”

$m = 1$



$m = 2$



$m = 4$



Detour – Min of I.I.D. Uniforms

If $Y_1, \dots, Y_m \sim \text{Unif}(0,1)$ (iid) where do we expect the points to end up?

$$\text{In general, } \mathbb{E}[\min(Y_1, \dots, Y_m)] = \frac{1}{m+1}$$

$$\mathbb{E}[\min(Y_1)] =$$

$$m = 1$$



$$\mathbb{E}[\min(Y_1, Y_2)] =$$

$$m = 2$$



$$\mathbb{E}[\min(Y_1, \dots, Y_4)] =$$

$$m = 4$$



Detour – Min of I.I.D. Uniforms

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$$\text{In general, } \mathbb{E}[\min(Y_1, \dots, Y_m)] = \frac{1}{m+1}$$

What is some intuition for this?

Detour – Min of I.I.D. Uniforms

If $Y_1, \dots, Y_m \sim \text{Unif}(0,1)$ (i.i.d.) where do we expect the points to end up?

e.g., what is $\mathbb{E}[\min\{Y_1, \dots, Y_m\}]$?

CDF: Observe that $\min\{Y_1, \dots, Y_m\} \geq y$ if and only if $Y_1 \geq y, \dots, Y_m \geq y$

Detour – Min of I.I.D. Uniforms

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$$\begin{aligned} P(\min\{Y_1, \dots, Y_m\} \geq y) &= P(Y_1 \geq y, \dots, Y_m \geq y) \\ y \in [0,1] &= P(Y_1 \geq y) \cdots P(Y_m \geq y) \quad (\text{Independence}) \\ &= (1 - y)^m \\ &\Rightarrow P(\min\{Y_1, \dots, Y_m\} \leq y) = 1 - (1 - y)^m \end{aligned}$$

$$F_Y(y) = P(\min\{Y_1, \dots, Y_m\} \leq y) = 1 - (1 - y)^m.$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = m(1 - y)^{m-1}.$$

$$\mathbb{E}[Y] = \int_0^1 y f_Y(y) dy = \int_0^1 y m(1 - y)^{m-1} dy = \frac{1}{m + 1}$$

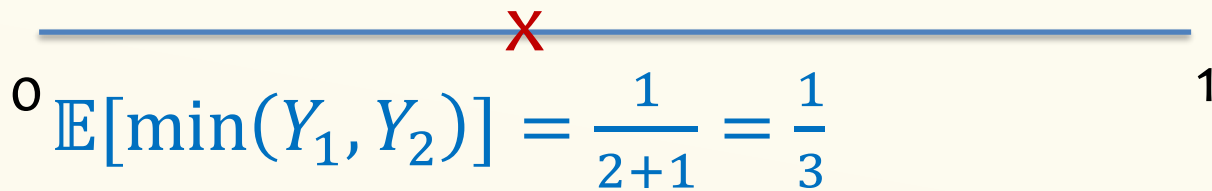
Detour – Min of I.I.D. Uniforms

If $Y_1, \dots, Y_m \sim \text{Unif}(0,1)$ (iid) where do we expect the points to end up?

$$\text{In general, } \mathbb{E}[\min(Y_1, \dots, Y_m)] = \frac{1}{m+1}$$

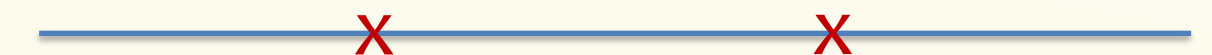
$$\mathbb{E}[\min(Y_1)] = \frac{1}{1+1} = \frac{1}{2}$$

$m = 1$



$$\mathbb{E}[\min(Y_1, Y_2)] = \frac{1}{2+1} = \frac{1}{3}$$

$m = 2$



$$\mathbb{E}[\min(Y_1, \dots, Y_4)] = \frac{1}{4+1} = \frac{1}{5}$$

$m = 4$



Back to counting distinct elements

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

N = # of IDs in the stream = 11, m = # of distinct IDs in the stream = 5

Want to compute number of **distinct** IDs in the stream, at least approximately.

How to do this without storing all the elements?

Distinct Elements – Hashing into $[0, 1]$

Hash function $h: U \rightarrow [0,1]$

Key property: Every time you hash the same element, you get the same value.

32, 12, 14, 32, 7, 12, 32, 7



$h(32)$, $h(12)$, $h(14)$, $h(32)$, $h(7)$, $h(12)$, $h(32)$, $h(7)$

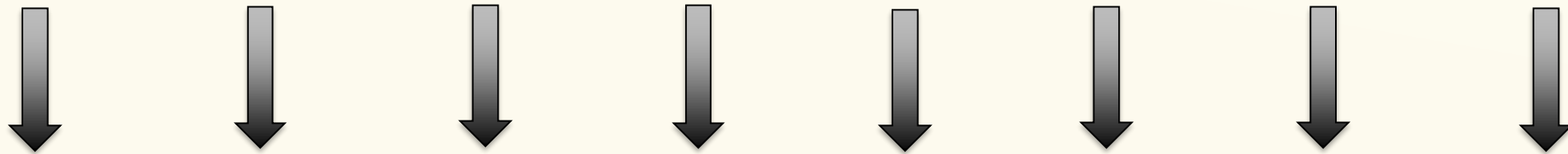
Distinct Elements – Hashing into $[0, 1]$

Hash function $h: U \rightarrow [0,1]$

Key Assumptions:

- Hash values are uniformly random.
- Distinct elements are hashed independently.

32, 12, 14, 32, 7, 12, 32, 7



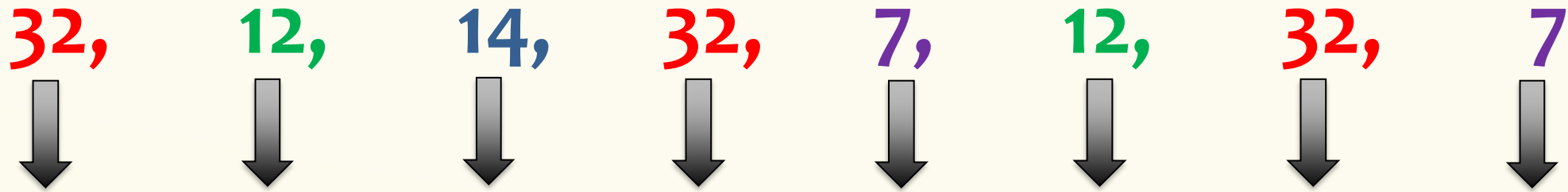
$h(32)$, $h(12)$, $h(14)$, $h(32)$, $h(7)$, $h(12)$, $h(32)$, $h(7)$

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32, 12, 14, 32, 7, 12, 32, 7



$h(32), h(12), h(14), h(32), h(7), h(12), h(32), h(7)$

Stream of length $N=8$, $m=4$ distinct elements

→ 4 i.i.d. RVs $h(32), h(12), h(14), h(7) \sim \text{Unif}(0,1)$

$$\rightarrow \mathbb{E}[\min\{h(32), h(12), h(14), h(7)\}] = \frac{1}{4+1} = \frac{1}{5}$$

Distinct Elements – Hashing into $[0, 1]$

Hash function $h: U \rightarrow [0,1]$

Key Assumptions:

- Hash values are uniformly random.
- Distinct elements are hashed independently.

x_1, x_2, \dots, x_N contains m distinct elements



$h(x_1), h(x_2), \dots, h(x_N)$ contains m i.i.d. rvs $\sim \text{Unif}(0,1)$
and $N - m$ repeats



$$\mathbb{E}[\min\{h(x_1), \dots, h(x_N)\}] = \frac{1}{m + 1}$$

A super duper clever idea!!!!

We can track $\min\{h(x_1), \dots, h(x_N)\}$ by storing only a single element.

Suppose that $\min\{h(x_1), \dots, h(x_N)\}$ is $\approx \mathbb{E}[\min\{h(x_1), \dots, h(x_N)\}]$.

Then we would be golden because

$$\mathbb{E}[\min\{h(x_1), \dots, h(x_N)\}] = \frac{1}{m+1}$$



A super duper clever idea!!!!

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Suppose that $\min\{h(x_1), \dots, h(x_N)\}$ is $\approx \mathbb{E}[\min\{h(x_1), \dots, h(x_N)\}]$.

Then we would be golden!

$$\mathbb{E}[\min\{h(x_1), \dots, h(x_N)\}] = \frac{1}{m+1}$$

$$\text{So } m = \frac{1}{\mathbb{E}[\min\{h(x_1), \dots, h(x_N)\}]} - 1$$



The MinHash Algorithm – Idea

$$m = \frac{1}{\mathbb{E}[\min\{h(x_1), \dots, h(x_N)\}]} - 1$$

1. Compute $\text{val} = \min\{h(x_1), \dots, h(x_N)\}$
2. Assume that $\text{val} \approx \mathbb{E}[\min\{h(x_1), \dots, h(x_N)\}]$
3. Output as estimate for m : $\text{round}\left(\frac{1}{\text{val}} - 1\right)$



The MinHash Algorithm – Implementation

Algorithm MinHash(x_1, x_2, \dots, x_N)

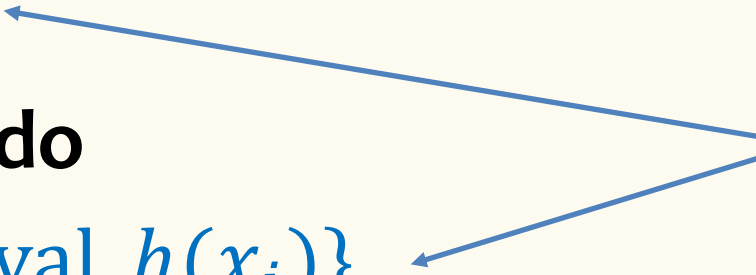
$\text{val} \leftarrow \infty$

for $i = 1$ **to** N **do**

$\text{val} \leftarrow \min\{\text{val}, h(x_i)\}$

return $\text{round}\left(\frac{1}{\text{val}} - 1\right)$

Memory cost = just remember val
(with sufficient precision)



MinHash Example

1. Compute $\text{val} = \min\{h(x_1), \dots, h(x_N)\}$
2. Assume that $\text{val} \approx \mathbb{E}[\min\{h(x_1), \dots, h(x_N)\}]$
3. Output $\text{round}\left(\frac{1}{\text{val}} - 1\right)$

Stream: 13, 25, 19, 25, 19, 19

Hashes: 0.51, 0.26, 0.79, 0.26, 0.79, 0.79

**What does
MinHash return?**

MinHash Example II

Stream: 11, 34, 89, 11, 89, 23

Hashes: 0.5, 0.21, 0.94, 0.5, 0.94, 0.1

Output is $\frac{1}{0.1} - 1 = 9$

Clearly, not a very good answer!

Not particularly unlikely: $P(h(x) < 0.1) = 0.1$

The MinHash Algorithm – Problem

Algorithm MinHash(x_1, x_2, \dots, x_N)

$\text{val} \leftarrow \infty$

for $i = 1$ **to** N **do**

$\text{val} \leftarrow \min\{\text{val}, h(x_i)\}$

return $\text{round}\left(\frac{1}{\text{val}} - 1\right)$

$\text{val} = \min\{h(x_1), \dots, h(x_N)\}$

$$\mathbb{E}[\text{val}] = \frac{1}{m+1}$$

Problem: val is not $\mathbb{E}[\text{val}]$!
How far is val from $\mathbb{E}[\text{val}]$?

$$\text{Var}(\text{val}) \approx \frac{1}{(m+1)^2}$$

How can we reduce the variance?

Idea: Repetition to reduce variance!

Use k independent hash functions h^1, h^2, \dots, h^k

$$\text{val}_1 = \min\{h^1(x_1), \dots, h^1(x_N)\}$$

$$\text{val}_2 = \min\{h^2(x_1), \dots, h^2(x_N)\}$$

...

$$\text{val}_k = \min\{h^k(x_1), \dots, h^k(x_N)\}$$

$$\widehat{\text{val}} \leftarrow \frac{1}{k} \sum_{i=1}^k \text{val}_i$$

Output as estimate

for m : $\text{round}(1/\widehat{\text{val}} - 1)$



How can we reduce the variance?

Idea: Repetition to reduce variance!

Use k independent hash functions h^1, h^2, \dots, h^k

Algorithm MinHash(x_1, x_2, \dots, x_N)

$val_1, \dots, val_k \leftarrow \infty$

for $i = 1$ **to** N **do**

for $j = 1$ **to** k **do** $val_j \leftarrow \min\{val_j, h^j(x_i)\}$

$\widehat{val} \leftarrow \frac{1}{k} \sum_{i=1}^k val_i$

return $\text{round}(1/\widehat{val} - 1)$



$$\text{Var}(\widehat{val}) = \frac{1}{k} \frac{1}{(m+1)^2}$$

How do we get a handle on how many hash functions to use?

CLT

Tail bounds

Tail Bounds (Idea)

Bounding the probability a random variable is far from its mean. Usually statements of the form:

$$\Pr(X \geq a) \leq b$$
$$\Pr(|X - E[X]| \geq a) \leq b$$

Useful tool when

- An approximation that is easy to compute is sufficient
- The process is too complex to analyze exactly

Key point

- A random variable might almost never be at least its expectation.
- Similarly, a random variable might almost always be at least its expectation or even much bigger.

Tail Bounds

Generally, the more you know about your random variable the better tail bounds you can get.

- Markov's Inequality 
- Chebyshev's Inequality

Markov's Inequality

Theorem. Let X be a random variable taking only non-negative values. Then, for any $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

$$\mathbb{P}(X \geq t \cdot \mathbb{E}(X)) \leq \frac{1}{t}.$$

Incredibly simplistic – only requires that the random variable is non-negative and only needs you to know expectation. You don't need to know **anything else** about the distribution of X .

Markov's Inequality – Proof

Theorem. Let X be a (discrete) random variable taking only non-negative values. Then, for any $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

$$\mathbb{E}(X) = \sum_{x \geq 0} x \cdot \mathbb{P}(X = x)$$

$$= \sum_{x \geq t} x \cdot \mathbb{P}(X = x) + \sum_{x < t} x \cdot \mathbb{P}(X = x)$$

Markov's Inequality – Proof

Theorem. Let X be a (discrete) random variable taking only non-negative values. Then, for any $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

$$\mathbb{E}(X) = \sum_x x \cdot \mathbb{P}(X = x)$$

$$= \sum_{x \geq t} x \cdot \mathbb{P}(X = x) + \sum_{x < t} x \cdot \mathbb{P}(X = x)$$

$$\geq \sum_{x \geq t} x \cdot \mathbb{P}(X = x)$$

$$\geq \sum_{x \geq t} t \cdot \mathbb{P}(X = x) = t \cdot \mathbb{P}(X \geq t)$$

≥ 0 because $x \geq 0$
whenever $\mathbb{P}(X = x)$
 ≥ 0 (takes only non-
negative values)

Follows by re-arranging terms
...

Example – Binomial Random Variable

Markov's inequality

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

Let X be Binomial RV with parameters. $n, p = 0.5$

$$\mathbb{E}(X) = \frac{n}{2}$$

What is the probability that $X \geq \frac{3n}{4}$?

Markov's inequality: $\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{3n} \cdot \frac{n}{2} = \frac{2}{3}$

Can we do better?

Some tail bounds

- Markov's Inequality
- Chebyshev's Inequality ◀

Using variance

- If we know more about the random variable, e.g. its variance, we can get a better bound!

Chebyshev's Inequality

Markov's inequality

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

Theorem. Let X be a random variable. Then, for any $t > 0$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq \frac{\text{Var}(X)}{t^2}.$$

Proof: Define $Z = X - \mathbb{E}(X)$

Definition of Variance

$$\mathbb{P}(|Z| \geq t) = \mathbb{P}(Z^2 \geq t^2) \leq \frac{\mathbb{E}(Z^2)}{t^2} = \frac{\text{Var}(X)}{t^2}$$

$|Z| \geq t$ iff $Z^2 \geq t^2$

Markov's inequality ($Z^2 \geq 0$)

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq \frac{\text{Var}(X)}{t^2}.$$

Example – Binomial Random Variable

Let X be Binomial RV with parameters. $n, p = 0.5$

$$\mathbb{E}(X) = \frac{n}{2}$$

$$\text{Var}(X) =$$

What is the probability that $X \geq \frac{3n}{4}$?

Chebychev's inequality: $\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq$

Markov's inequality: $\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{3n} \cdot \frac{n}{2} = \frac{2}{3}$

Tail Bounds

Useful for approximations of complex systems. How good the approximation is depends on the actual distribution and the context you are using it in.

- Usually loose upper-bounds are okay when designing for worst-case

Generally, the more you know about your random variable the better tail bounds you can get.

And one more thing....

Detour – Min of I.I.D. Uniforms

Useful fact. For any random variable Y taking non-negative values

$$\mathbb{E}[Y] = \int_0^{\infty} P(Y \geq y) dy$$

Proof

$$\begin{aligned} \mathbb{E}[Y] &= \int_0^{\infty} x \cdot f_Y(x) dx = \int_0^{\infty} \left(\int_0^x 1 dy \right) \cdot f_Y(x) dx = \int_0^{\infty} \int_0^x f_Y(x) dy dx \\ &= \int_0^{\infty} \int_y^{\infty} f_Y(x) dx dy = \int_0^{\infty} P(Y \geq y) dy \end{aligned}$$

Detour – Min of I.I.D. Uniforms

$Y_1, \dots, Y_m \sim \text{Unif}(0,1)$ (i.i.d.)

$Y = \min\{Y_1, \dots, Y_m\}$

Useful fact. For any continuous random variable Y taking non-negative values

$$\mathbb{E}[Y] = \int_0^{\infty} P(Y \geq y) dy$$

$$\mathbb{E}[Y] = \int_0^{\infty} P(Y \geq y) dy = \int_0^1 (1 - y)^m dy$$

$$= -\frac{1}{m+1} (1 - y)^{m+1} \Big|_0^1 = 0 - \left(-\frac{1}{m+1} \right) = \frac{1}{m+1}$$

Useful fact. For any discrete random variable Y taking non-negative values $\sum_{k=0}^{\infty} P(Y > k)$

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} P(Y > k)$$