Midterm review

CSE 312 Spring 26

Lecture 20

Midterm Thursday at 6pm (Try to come 10 mins early)

- Last name starts with A-P → Bagley 131
- Last name starts with Q-Z → Johnson Hall 102
- All special circumstances have been dealt with.

- **Bring your photo ID with you!**

- **We will be checking IDs once you have sat down – in some cases during the exam.**

NO CLASS FRIDAY – enjoy a mini-break

Some other minor notes on exam

- When a calculation is simple, I asked you to provide the final answer.

  e.g., 25 + 100 + 10 + 20.

- If the calculation was slightly more complex, okay to leave it:

  e.g., 0.8*0.32 + 5.82*25 + 0.72 * 0.05 + 29.6

- The probability of some event is 5% – that means it occurs with probability 0.05
- Go over the cheat sheet so you can easily find what you’re looking for during the test.

Cheat Sheet

The Sum Rule: If an experiment can either end up being one of N outcomes, or one of M outcomes (where there is no overlap), then the total number of possible outcomes is: $N + M$.

The Product Rule: If an experiment has N_1 outcomes for the first stage, N_2 outcomes for the second stage, ..., and N_m outcomes for the m^{th} stage, then the total number of outcomes of the experiment is $N_1 \times N_2 \cdot \cdots \cdot N_m = \prod_{i=1}^{m} N_i$.

Complementary Counting: Let $\mathcal{U}$ be a (finite) universal set, and S a subset of interest. Then, $\mid S \mid = \mid \mathcal{U} \mid - \mid \mathcal{U} \setminus S \mid$.

Permutation: The number of orderings of N **distinct** objects is $N! = N \cdot (N-1) \cdot (N-2) \cdot \ldots 3 \cdot 2 \cdot 1$.

k-Permutations: If we want to *pick* (**order matters**) only k out of n distinct objects, the number of ways to do so is:

$$P(n,k) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

Multinomial Coefficients: If we have k distinct types of objects (n total), with n_1 of the first type, n_2 of the second, ..., and n_k of the k-th, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \ldots n_k!}$$

k-Combinations/Binomial Coefficients: If we want to *choose* (**order doesn't matter**) only k out of n distinct objects, the number of ways to do so is:

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

Encoding/Stars and Bars Method: The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Binomial Theorem: Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Principle of Inclusion-Exclusion (PIE):
2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
k events: singles - doubles + triples - quads + ...

Pigeonhole Principle: If there are n pigeons we want to put into k holes (where $n > k$), then at least one pigeonhole must contain at least 2 (or to be precise, $\lceil n/k \rceil$) pigeons.

Combinatorial Proofs: To prove two quantities are equal, you can come up with a combinatorial situation, and show that both in fact count the same thing, and hence must be equal.

Key Probability Definitions: The **sample space** is the set Ω of all possible outcomes of an experiment. An **event** is any subset $E \subseteq \Omega$. Events E and F are **mutually exclusive** if $E \cap F = \varnothing$.

Probability space: A *probablity space* is a pair $(\Omega, \mathbb{P})$, where Ω is the sample space and $\mathbb{P} : \Omega \to [0,1]$ is a *probability measure* such that $\sum_{x \in \Omega} \mathbb{P}(x) = 1$. The probability of an event $E \subseteq \Omega$ is $\mathbb{P}(E) = \sum_{x \in E} \mathbb{P}(x)$.

Equally Likely Outcomes: If Ω is a sample space such that each of the unique outcome elements in Ω are equally likely, then for any event $E \subseteq \Omega$: $\mathbb{P}(E) = |E|/|\Omega|$.

Conditional Probability: $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Bayes Theorem: $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}$

Partition: Non-empty events $E_1, \ldots, E_n$ **partition** the sample space Ω if they are both:

- **(Exhaustive)** $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$ (they cover the entire sample space).
- **(Pairwise Mutually Exclusive)** For all $i \neq j$, $E_i \cap E_j = \varnothing$ (none of them overlap)

Note that for any event E, E and E^C always form a partition of Ω.

Law of Total Probability (LTP): If events $E_1, \ldots, E_n$ partition Ω, then for any event F:

$$\mathbb{P}(F) = \sum_{i=1}^{n} \mathbb{P}(F \cap E_i) = \sum_{i=1}^{n} \mathbb{P}(F \mid E_i)\mathbb{P}(E_i)$$

Bayes Theorem with LTP: : Suppose $A_1, \ldots, A_n$ partition Ω and let B be any event. Then $\mathbb{P}(A_1|B) = \dfrac{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1)}{\sum_{i=1}^{n} \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)}$. In particular, $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B \mid A)\mathbb{P}(A) + \mathbb{P}(B \mid A^C)\mathbb{P}(A^C)}$

Chain Rule: Let $A_1, \ldots, A_n$ be events with nonzero probabilities. Then:
$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdots \mathbb{P}(A_n \mid A_1 \cap \cdots \cap A_{n-1})$$

Independence: A and B are **independent** if any of the following equivalent statements hold:

1. $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$ 2. $\mathbb{P}(A \mid B) = \mathbb{P}(A)$ 3. $\mathbb{P}(B \mid A) = \mathbb{P}(B)$

Mutual Independence: We say n events $A_1, A_2, \ldots, A_n$ are **(mutually) independent** if, for *any* subset $I \subseteq [n] = \{1, 2, \ldots, n\}$, we have
$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i)$$
This equation is actually representing 2^n equations since there are 2^n subsets of $[n]$.

Conditional Independence: A and B are **conditionally independent given an event** C if any of the following equivalent statements hold:

1. $\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)$ 2. $\mathbb{P}(A \mid B \cap C) = \mathbb{P}(A \mid C)$ 3. $\mathbb{P}(B \mid A \cap C) = \mathbb{P}(B \mid C)$

Random Variable (RV): A random variable (RV) X is a numeric function of the outcome $X : \Omega \to \mathbb{R}$.
The set of possible values X can take on is its **range/support**, denoted Ω_X.
If Ω_X is finite or countable infinite (typically integers or a subset), X is a **discrete RV**.

Probability Mass Function (PMF): For a discrete RV X, assigns probabilities to values in its range. That is $p_X : \Omega_X \to [0, 1]$ such that: (1) $p_X(k) \geqslant 0$ for all $k \in \Omega_X$; (2) $\sum_{k \in \Omega_X} p_X(k) = 1$.
Furthermore, $p_X(k) = \mathbb{P}(X = k)$.

Cumulative Distribution Function (CDF): The **cumulative distribution function (CDF)** of ANY random variable (discrete or continuous) is defined to be the function $F_X : \mathbb{R} \to \mathbb{R}$ with $F_X(t) = \mathbb{P}(X \leqslant t)$.

Independence of RVs (Discrete): Discrete RVs X, Y are **independent**, written $X \perp Y$, if for all $x \in \Omega_X$ and $y \in \Omega_Y$:
$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$.

Expectation (Discrete): The **expectation** of a discrete RV X is:
$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$.

Law of the Unconscious Statistician (LOTUS): For a RV X and function g: If X is discrete, $\mathbb{E}[g(X)] = \sum_{b \in \Omega_X} g(b) \cdot p_X(b)$.

Linearity of Expectation (LoE): For any random variables X, Y (possibly dependent):
$\mathbb{E}[aX + bY + c] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y] + c$

Multiplicativity of expectation: For any independent random variables X, Y:
$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Linearity of Expectation with Indicators: If asked only about the expectation of a RV X which is some sort of "count" (and not its PMF), then you may be able to write X as the sum of possibly dependent **indicator** RVs $X_1, \ldots, X_n$, and apply LoE, where for an indicator RV X_i, $\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1)$.

Variance: $\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$.

Standard Deviation (SD): $\sigma_X = \sqrt{\text{Var}(X)}$.

Property of Variance: $\text{Var}(aX + b) = a^2 \text{Var}(X)$.

Variance Adds for Independent RVs: If X, Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

Student name: ________________ **Student Number:** ________________

CSE 312: Foundations of Computing II - Spring 2026

Midterm A

Important: Do not turn the page until instructed to start. In the meantime, read the instructions on this page carefully.

Instructions. You have 90 minutes to complete this midterm (though the exam has been designed for a 50 minute slot.)

- **Write your name and student number on top of this page.**
- **If you do not have an ID with a picture, you will not be able to take the exam.**
- This is a **closed-book, closed notes exam** with the exception of the cheat sheet that we are giving you.
- **No electronics are allowed** during the exam (no smart phones, no laptops, no smart watches, no pocket calculators, etc). **Before you come into the room** where you are taking the test, store them in your bag/backpack and do not take them out until you leave the room after the exam. If we see any such items once you are in the room where you take the test, we will take your test away.
- Write your final solutions in the appropriate boxes. Be sure that what you put in the box is **only** your final answer and is written neatly.
- If your final answer to a question is correct, you will get full credit *regardless of whether or not you provide any explanation*. On some of the problems (**but not all**), we will provide a little bit of partial credit if you provide correct partial explanations. There will be no partial credit on any True/False or multiple choice problem.
- If a problem looks difficult, I recommend moving on to another problem and coming back later.

Good luck!

Task 1 – Short final answer / True-False / Multiple Choice [54 pts]

(6 points each)

No partial credit on any of the parts of this problem. Also, be sure to write your final final answer in the box provided (except for multiple choice/true-false questions). You do **not** need to simplify answers involving numbers unless otherwise specified.

a) Put your final answer in the box: Something something

Final final answer: ☐

b) Put your final answer in the box: Something something. Your answer should be a single number.

Final final answer: ☐

c) Fill in the correct circle. Something something something

- ◯ True
- ◯ False

d) Fill in the correct circle. Something something something

◯ True

◯ False

e) Multiple Choice (Fill in the correct circle): Something something

◯ first choice

◯ second choice

◯ third choice

◯ fourth choice

Task 2 – Another question [22 pts]

Some setup.

You do NOT need to explain your final answers (though you will not get any partial credit without explanations). Also, you do NOT need to simplify your final answers, but please **write each final final answer in the box provided.**

a) (8 points) **Put your final answer in the box:** Question

Final final answer:

b) (6 points) **Fill in the appropriate circle:** Something something

◯ True

◯ False

c) (8 points) **Put your final answer in the box:** Something something

Final final answer:

Task 3 – Another question [20 pts]

Setup for problem

You do NOT need to explain your final answers (though you will not get any partial credit without explanations). Also, you do NOT need to simplify your final answers, but please **write each final final answer in the box provided.**

For the purposes of this problem, let A,B, C be events or random variables denoting something or other. **No partial credit can be offered unless you use these event or random variable names in your solution.**

a) (10 points) Something something

Final final answer:

b) (10 points) Something something

Let p be the final answer to part a). If you need to use your final answer from part a), please use p instead of your final answer. Your final final answer should involve numbers and the quantity p only, but you do not need to simplify it.

Final final answer:

- Suppose that Y is Binomial with parameters 100 and 0.2.
- Let X be the number of heads if we toss a coin with probability p of coming up Heads independently Y times. What is the probability that $X = k$?

A professor has a test bank of 20 questions that she will draw on for a particular exam. A particular student knows how to solve 12 of them. The exam she gives is a random subset of 8 of the questions.

- What is the probability that the student knows how to solve all 8 problems?
- What is the probability that the student knows how to solve exactly 6 of the problems?
- What is the expected number of questions the student will get right?

- Describe the probability mass function of a discrete distribution with mean 10 and variance 9 that takes only 2 distinct values.

- Let Z be a random variable. If $Var(2Z+5) = E(3Z^2) = 12$ and Z is nonnegative, then what is $E(Z)$?

- What is the conditional probability that a random 5-card poker hand is a 4 of a kind (i.e., contains 4 cards of 1 rank and 1 card of a different rank) given that it contains at least one pair?

True or False

- For any events E and F s.t. $Pr(E \mid E \cap F) > 0$, it holds that $Pr(E \mid E \cap F) \leq \Pr(E|F)$

- N voters in a certain country are voting in an election between k candidates: A_1, A_2, … , A_k. Suppose that independently each person votes for candidate A_i with probability p_i, where $\sum_{i=1}^{k} p_i = 1$.
- Let X be the number of votes for either A_1 or A_2 . What distribution from our zoo is X and what are its parameters?

- Same setup. Let X_1 be the number of votes A_1 gets and let X_2 be the number of votes A_2 gets. Are X_1 and X_2 independent?

- Same setup. Let X_i be the number of votes A_i gets and let Z be the number of distinct candidates that get exactly 10 votes. What is $\mathrm{E}(Z)$?

True or False

- If X and Y are independent random variables, then so are 5x+3, and 7Y-2.

A DNA sequence can be thought of as a string made up of 4 bases:

A, T, G, C

Suppose that the DNA sequence is random: the base in each position is selected independently of other positions, and for each particular position, one of the 4 bases is selected such that the letters G and C occur with probability 0.2 each and A and T occur with probability 0.3 each.

In a sequence of length n, what is the expected number of occurrences of the sequence AATGTC?