We will use pollev today.

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# More Counting

CSE 312 Spring 26

Lecture 2

# Questions, etc.

We will use pollev today.

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- About syllabus?
- Anything else?

**Midterm on Thursday, May 14 from 6- 7:30pm**

Full Pset 1 out this evening.

Take note of readings linked from the schedule page.

# Quick Recap

- Sum Rule

  If you can choose from

  - Either one of n options,
  - OR one of m options with NO overlap with the previous n,

  then the number of possible outcomes of the experiment is n+m
- Product Rule

  In a sequential process, if there are

  - $n_1$ choices for the first step,
  - $n_2$ choices for the second step (given the first choice), … , and
  - $n_k$ choices for the k step (given the previous choices),

  then the total number of outcomes is $n_1 \times n_2 \times \ldots \times n_k$
- Complementary Counting

# Number of Subsets – Not the same as k-permutations.

*"How many size-5 **subsets** of* $\{A, B, \ldots, Z\}$*?"*

E.g., {A,Z,U,R,E}, {B,I,N,G,O}, {T,A,N,G,O}. But not: {S,T,E,V}, {S,A,R,H},…

Different from $k$-permutations: **NO ORDER**

Different sequences: TANGO, OGNAT, ATNGO, NATGO, ONATG …

Same set: {T,A,N,G,O}, {O,G,N,A,T}, {A,T,N,G,O}, {N,A,T,G,O}, {O,N,A,T,G}… …

# Number of Subsets of size 5 from alphabet of 26

Consider the following process:

1. Choose an **unordered** subset $S \subseteq \{A, B, \ldots, Z\}$ of size $|S| = 5$. e.g. $S = \{A, G, N, O, T\}$

2. Choose a permutation of letters in $S$

   e.g., *TANGO, AGNOT, NAGOT, GOTAN, GOATN, NGOAT, …*

Outcome: An **ordered** sequence of 5 distinct letters from $\{A, B, \ldots, Z\}$

$$??? \times 5! = \frac{26!}{21!}$$

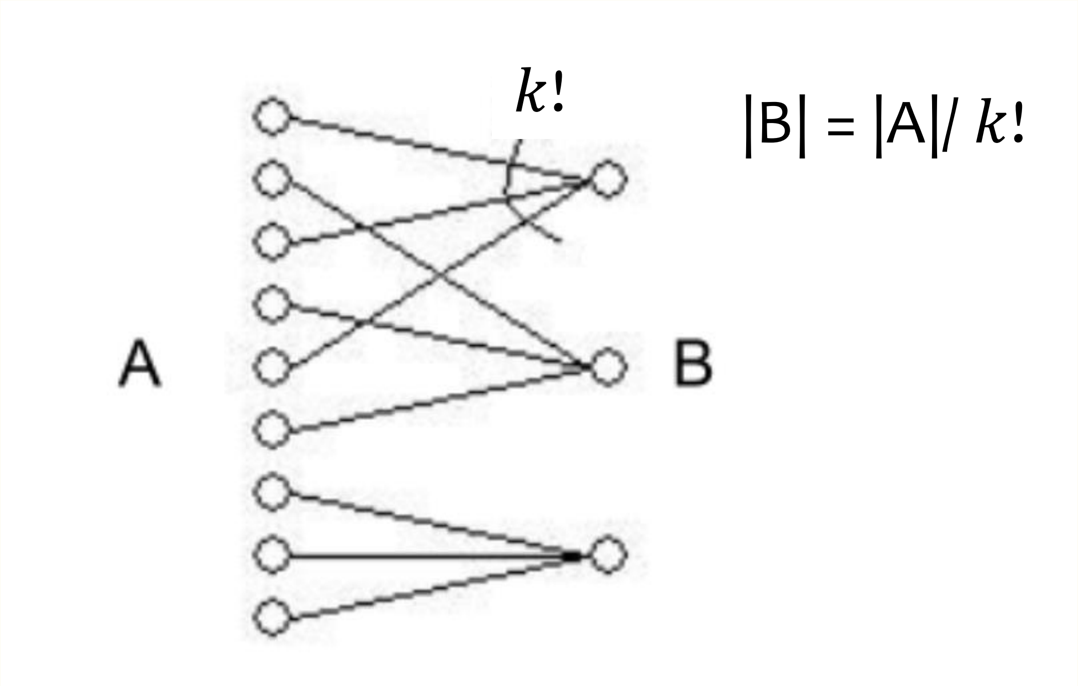
$$??? = \frac{26!}{21!\,5!}$$

## Count the number of unordered subsets of size $k$ chosen from a set of $n$ distinct elements.

Each element of A is an ordered set of $k$ out of $n$ distinct elements

Each element of B is an unordered set of $k$ out of $n$ distinct elements

$$|A| = \frac{n!}{(n-k)!}$$




Therefore, $|B| = \frac{n!}{k!(n-k)!}$

# Number of Subsets -- don't care about order

**Fact.** The number of subsets of size $k$ of a set of size $n$ is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

we say "$n$ choose $k$"

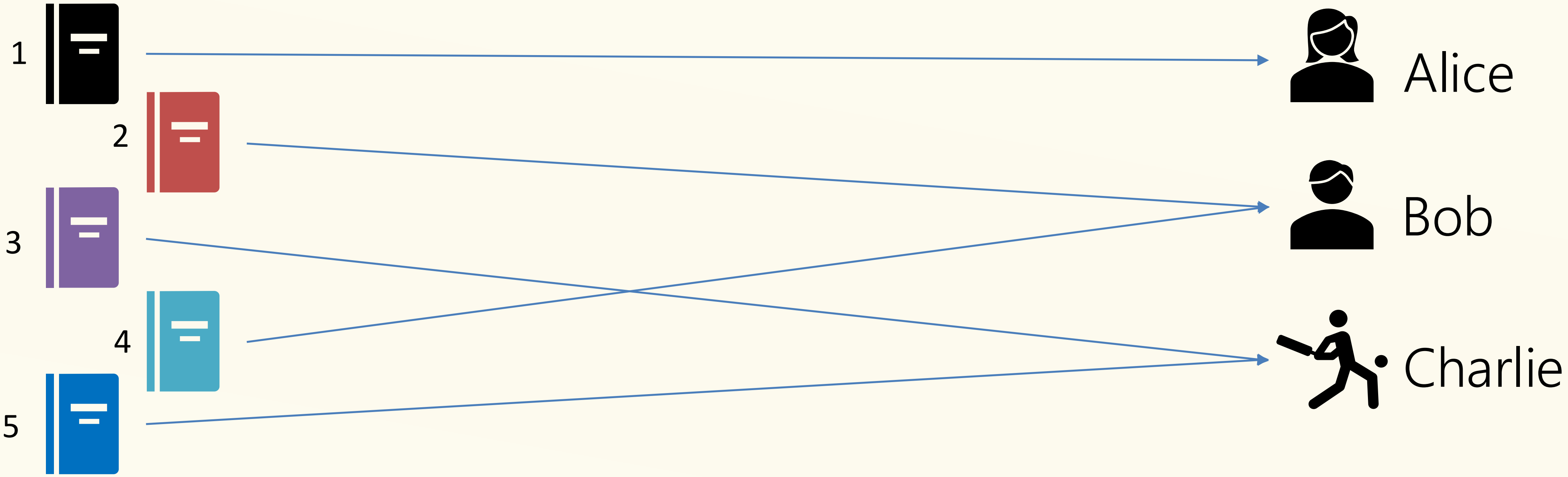
[also called **combinations** or **binomial coefficients**]

# Summary so far

- k-sequences: How many length k sequences over alphabet of size n? repetition allowed.
  - Product rule ➔ $n^k$
- k-permutations: How many length k sequences over alphabet of size n, without repetition?
  - Permutation ➔ $\frac{n!}{(n-k)!}$
- k-combinations: How many size k subsets of a set of n distinct elements (without repetition and without order)?
  - Combination ➔ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

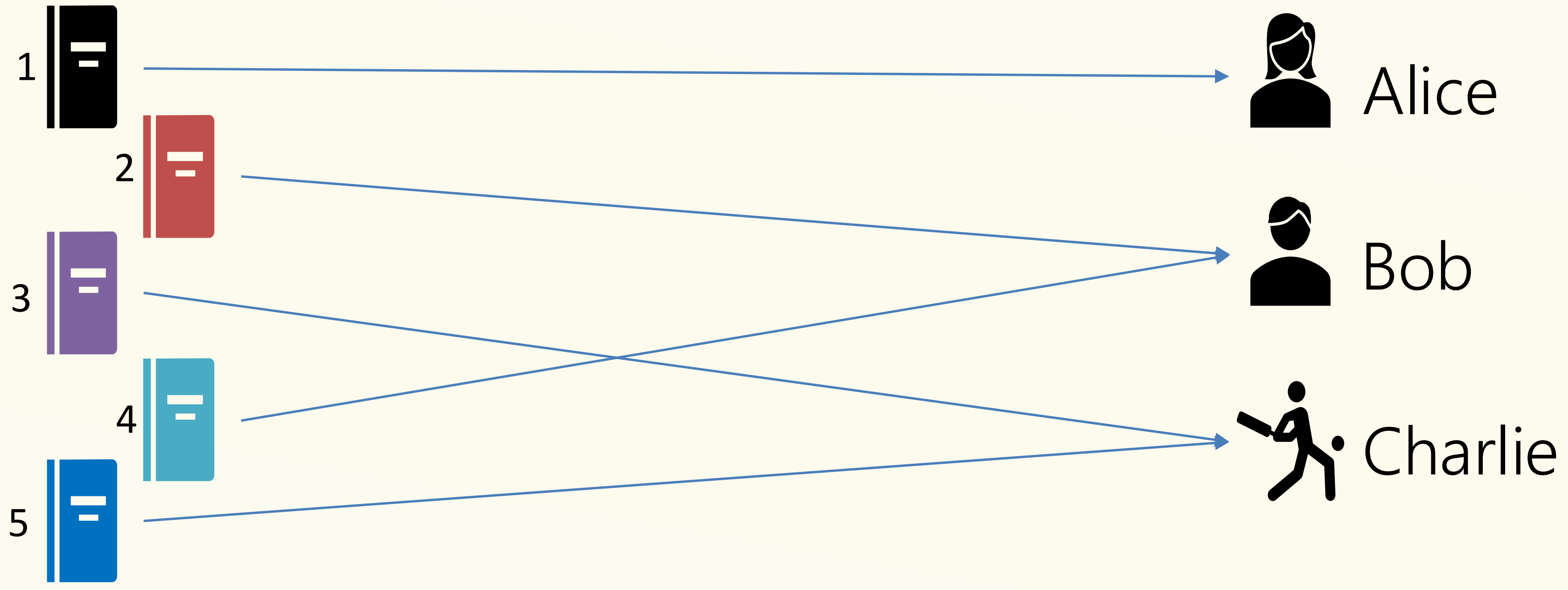
# Assigning Books

- We have 5 books to split among 3 people (Alice, Bob, and Charlie)
- Every book goes to exactly one person, but each person could end up with no books (or all of them, or something in between).



# Assigning Books – first try

- We have 5 books to split among 3 people (Alice, Bob, and Charlie)
- Every book goes to exactly one person, but each person could end up with no books (or all of them, or something in between).
- **First try:** apply product rule, count number of possibilities for how many books Alice gets, then number of possibilities for how many books Bob gets, then number of possibilities for how many books Charlie gets




***$2^5$ = 32 choices***

***$2^5$ = 32 choices***

***$2^5$ = 32 choices***

***Product =$2^{15}$***

# Assigning Books (first try): correct, overcount or undercount?

- Attempt 1: We're choosing subsets!
  - Alice could get any of the $2^5=32$ subsets of the books.
  - Bob could get any of the $2^5=32$ subsets of the books.
  - Charlie could get any of the $2^5=32$ subsets of the books.
  - Total is product of those three steps $2^5 \times 2^5 \times 2^5 = 2^{15}$

We overcounted!

*If Alice gets {1,2}, Bob can't get any of the possible subsets, he can only get a subset of {3,4,5} And Charlie's subset is just whatever is leftover after Alice and Bob get theirs…*

# Assigning Books – second try

- We have 5 books to split among 3 people (Alice, Bob, and Charlie)
- Every book goes to exactly one person, but each person could end up with no books (or all of them, or something in between).
- Apply product rule the other way: count number of possibilities for who gets book 1, then number of possibilities for who gets book 2, etc.

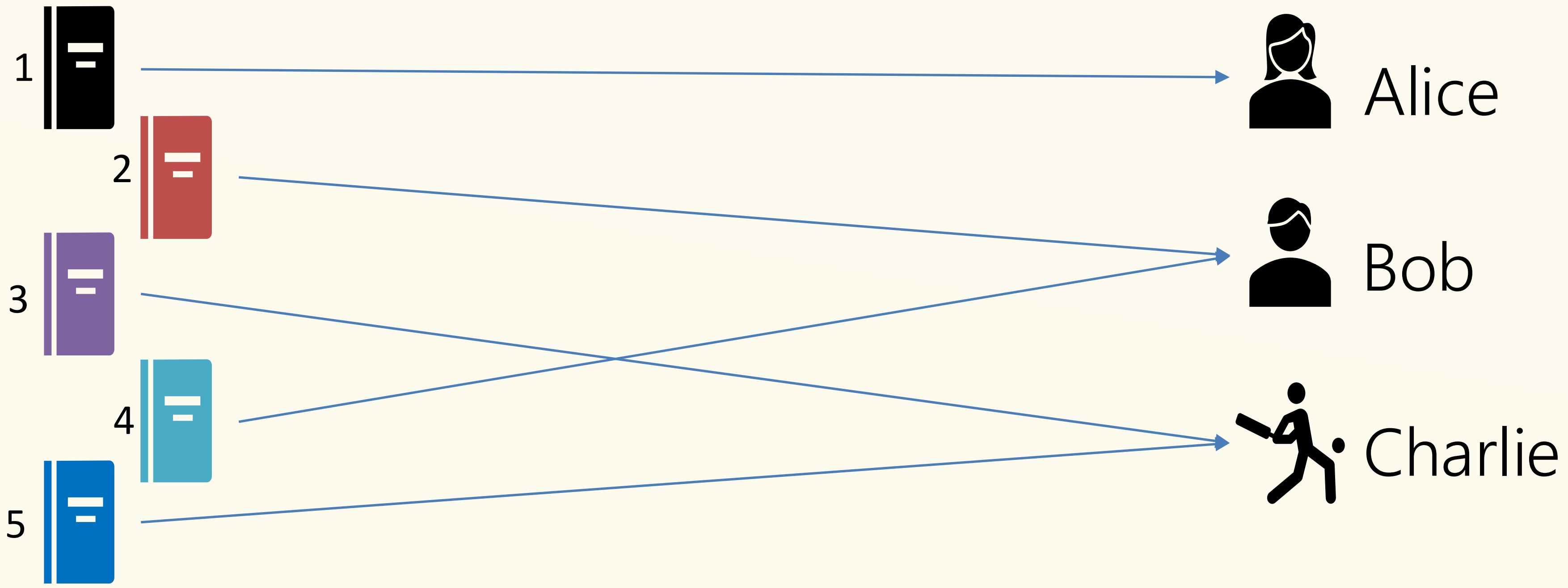
***3 choices***

***3 choices***

***3 choices***

***3 choices***

***3 choices***



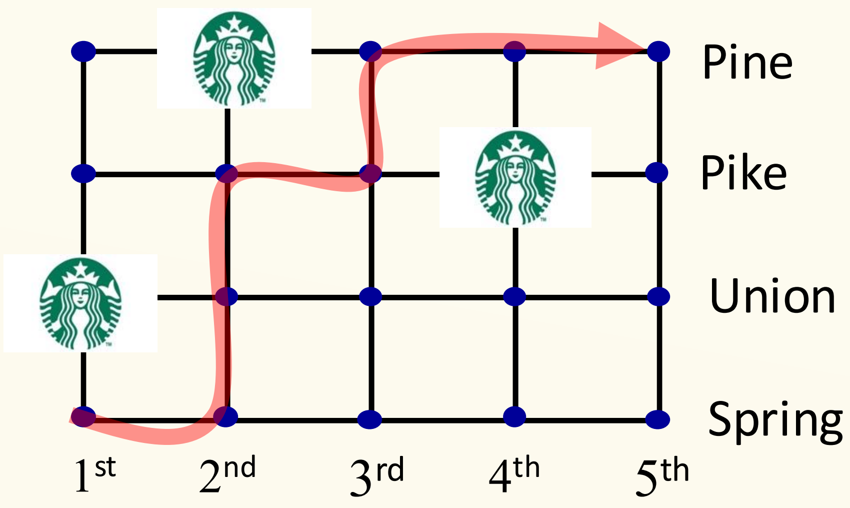

***Product = $3^5$***

# Lesson: Representation of what we are counting is very important!

Think about the various possible ways you could make a sequence of choices that leads to an outcome in the set of outcomes you are trying to count.

# Counting Paths

How many ways are there to walk from 1st and Spring to 5th and Pine only travelling north or east?




E N N E N E E

# Counting Paths - possible answers?

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How many ways are there to walk from 1st and Spring to 5th and Pine only travelling north or east?

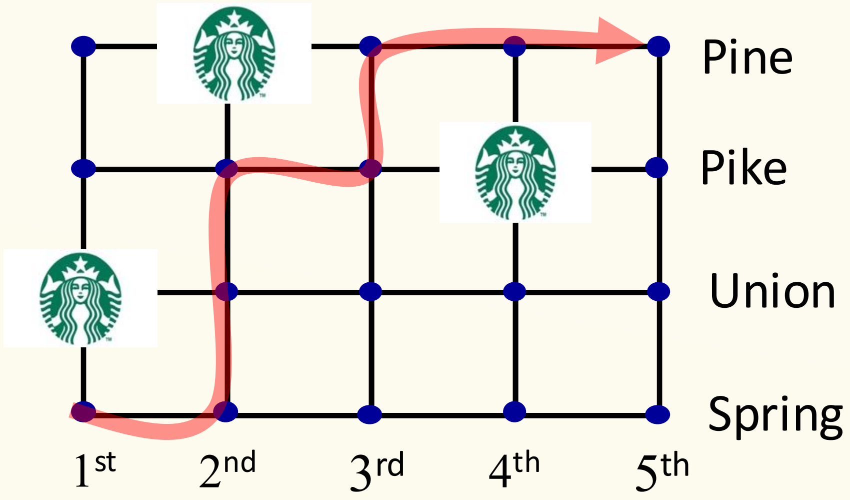
Poll:

A. $2^7$

B. $\frac{7!}{4!}$

C. $\binom{7}{4} = \frac{7!}{4!3!}$

D. $\binom{7}{3} = \frac{7!}{3!4!}$




E N N E N E E

# Counting Paths - answers?

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How many ways are there to walk from 1st and Spring to 5th and Pine only travelling north or east?

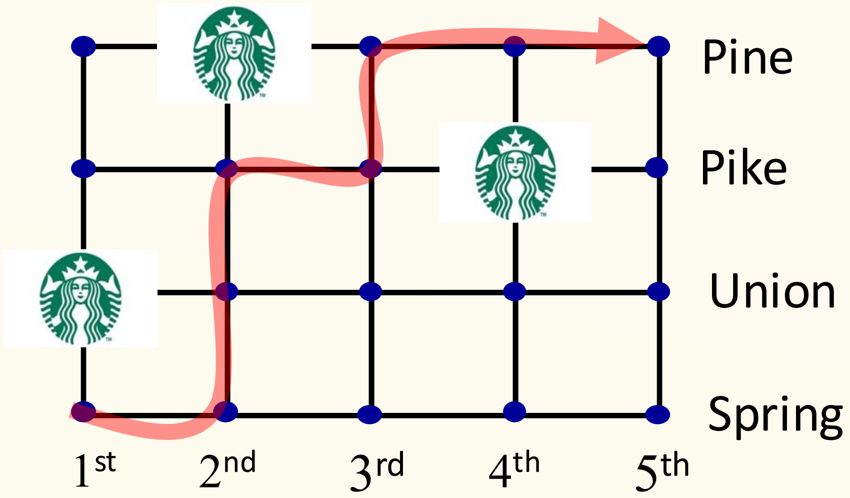
Poll:

*A.* $2^7$

*B.* $\frac{7!}{4!}$

✅ *C.* $\binom{7}{4} = \frac{7!}{4!3!}$

✅ *D.* $\binom{7}{3} = \frac{7!}{3!4!}$




E N N E N E E

# Symmetry in binomial coefficients

**Fact.** The number of subsets of size $k$ of a set of size $n$ is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Fact: $\binom{n}{k} = \binom{n}{n-k}$

# Symmetry in binomial coefficients – first proof

**Fact.** $\binom{n}{k} = \binom{n}{n-k}$

**Proof.** $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$

This is called an algebraic proof.

Proof by checking algebra.

**But what's the intuition?**

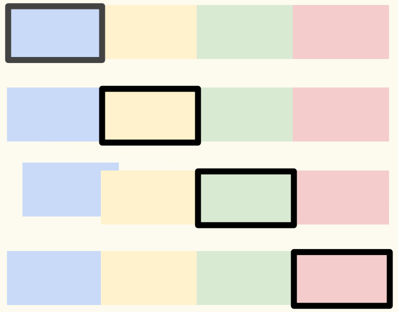
# Symmetry in binomial coefficients – a more intuitive proof

**Fact.** $\binom{n}{k} = \binom{n}{n-k}$

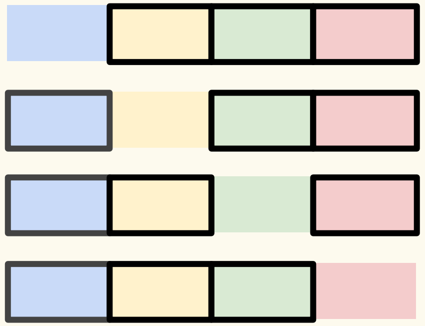
Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which n-k elements are excluded

4 choose 1 to include.



4 choose 3 to exclude



# Symmetry in binomial coefficients – summary of combinatorial proof

**Fact.** $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which n-k elements are excluded

This is called a **combinatorial argument/proof**

Let S be a set of objects

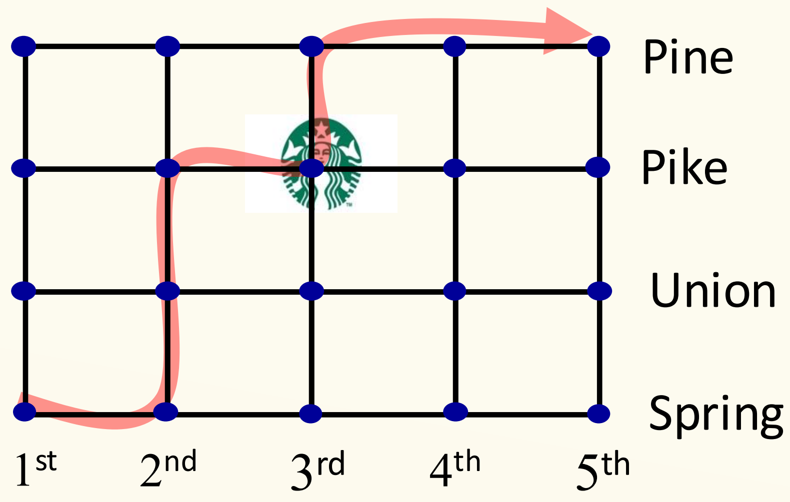
– Show how to count |S| one way => |S| = N

– Show how to count |S| another way => |S| = m

Conclude that N = m

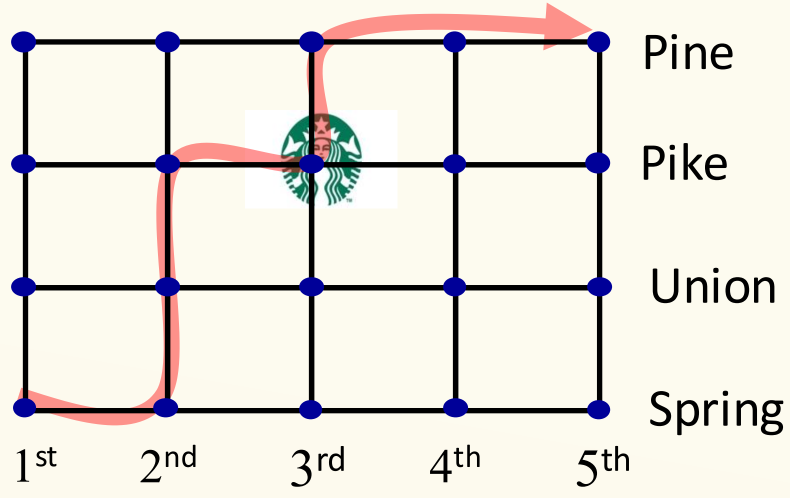
# Counting Paths – with stop at Starbucks

How many ways are there to walk from 1st and Spring to 5th and Pine only travelling north or east if you want to also stop at Starbucks on 3rd and Pike?



# Counting Paths with stop at Starbucks -- possible answers

How many ways are there to walk from 1$^{st}$ and Spring to 5$^{th}$ and Pine only travelling north or east if you want to also stop at Starbucks on 3$^{rd}$ and Pike?




Poll:

A. $\binom{7}{3}$

B. $\binom{7}{3}\binom{7}{1}$

C. $\binom{4}{2}\binom{3}{1}$

D. $\binom{4}{2}\binom{3}{2}$

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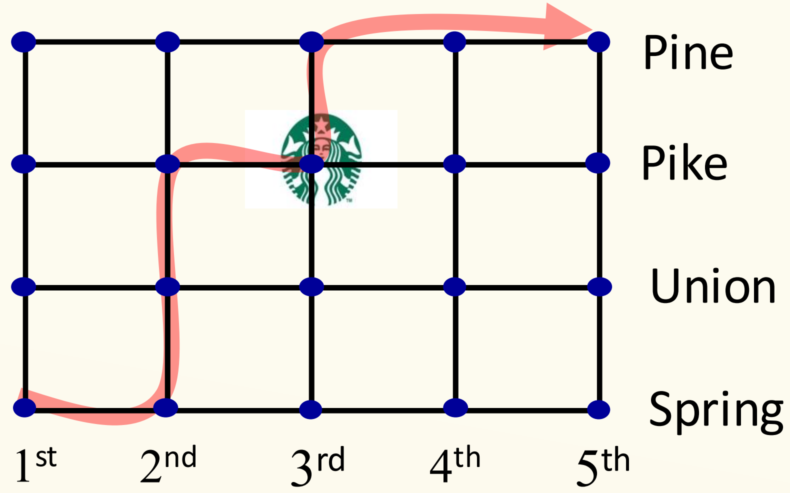
# Counting Paths with stop at Starbucks -- answers

How many ways are there to walk from $1^{st}$ and Spring to $5^{th}$ and Pine only travelling north or east if you want to also stop at Starbucks on $3^{rd}$ and Pike?




Poll:

A. $\binom{7}{3}$

B. $\binom{7}{3}\binom{7}{1}$

✓ C. $\binom{4}{2}\binom{3}{1}$

✓ D. $\binom{4}{2}\binom{3}{2}$

# Agenda (part 2)

- Recap & Examples
- **Binomial Theorem**
- Multinomial Coefficients
- Inclusion-Exclusion
- Combinatorial Proofs

# Binomial Theorem: Idea

$(x + y)^2 = (x+y)(x+y)$

$= xx + xy + yx + yy$

$= x^2 + 2xy + y^2$

$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$

$= xxxx + yyyy + xyxy + yxyy + \ldots$

# Binomial Theorem: Let's figure out one coefficient

$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$

$= xxxx + yyyy + xyxy + yxyy + \ldots$

What is the coefficient of $xy^3$?

*A.* 4

*B.* $\binom{4}{1}$

*C.* $\binom{4}{3}$

*D.* 3

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# Binomial Theorem: Another example coefficient

$$(x+y)^n = (x+y)(x+y)(x+y)\ldots.(x+y)$$

Each term is of the form $x^k y^{n-k}$, since each term is is made by multiplying exactly n variables, either x or y.

How many times do we get $x^k y^{n-k}$ ?

# Binomial Theorem: coefficients of C(n,k)

$$(x+y)^n = (x+y)(x+y)(x+y)\ldots.(x+y)$$

Each term is of the form $x^k y^{n-k}$, since each term is is made by multiplying exactly n variables, either x or y.

How many times do we get $x^k y^{n-k}$ ?

The number of ways to choose k of the n “x variables” we multiply together. The rest will be y’s.

$$\binom{n}{k} = \binom{n}{n-k}$$

# Binomial Theorem

**Theorem.** Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

# Binomial Theorem – a corollary

**Theorem.** Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

**Corollary**

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:
Substitute $x$ =1, $y$ =1 in Binomial Theorem

# Agenda (part 3)

- Recap & Examples
- Binomial Theorem
- **Multinomial Coefficients**
- Inclusion-Exclusion
- Combinatorial Proofs

# Permutations of word MATH

How many ways to rearrange the letters in the word "MATH"?

<u>Poll:</u>

*A.* $\binom{26}{4}$

*B.* $4^4$

*C.* $4!$

D. *I don't know*

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# Permutations of word MATH (answer)

How many ways to rearrange the letters in the word ”MATH”?

Poll:

A. $\binom{26}{4}$

B. $4^4$

C. $4!$ ✓

D. *I don’t know*

# Permutations of word MUUMUU

How many ways to rearrange the letters in the word ”MUUMUU”?

# Example 2 – Word Permutations

How many ways to rearrange the letters in the word "MUUMUU"?

- Choose where the 2 M's go, and then the U's are set **OR**
- Choose where the 4 U's go, and then the M's are set

Either way, we get $\binom{6}{2} \cdot \binom{4}{4} = \binom{6}{4} \cdot \binom{2}{2} = \frac{6!}{2!4!}$

# Example 2 – Word Permutations (another way to count)

How many ways to rearrange the letters in the word ”MUUMUU”?

Arrange the 6 letters as if they were distinct.

$$M_1U_1U_2M_2U_3U_4$$

Then divide by 4! to account for duplicate M’s and divide by 2! to account for duplicate U’s.

Yields $\frac{6!}{2!4!}$

# Example 3 – Word Permutations

How many ways to rearrange the letters in the word ”GODOGGY”?

Poll:

*A.* $7!$

*B.* $\frac{7!}{3!}$

*C.* $\frac{7!}{3!2!1!1!}$

*D.* $\binom{7}{3} \cdot \binom{4}{2} \cdot 2!$

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# Example 3 – Word Permutations (answer)

How many ways to rearrange the letters in the word "GODOGGY"?
(3 G's, 2 O's, 1 D, 1 Y)

Poll:

A. $7!$

B. $\frac{7!}{3!}$

✅ C. $\frac{7!}{3!2!1!1!}$

✅ D. $\binom{7}{3} \cdot \binom{4}{2} \cdot 2!$

# Multinomial coefficients

If we have $k$ types of objects, with $n_1$ of the first type, $n_2$ of the second type, … , $n_k$ of the k^th^ type, where $n = n_1 + n_2 + \cdots + n_k$ then the number of arrangements of the $n$ objects is

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k\,!}$$

Note that objects of the same type are indistinguishable.

# Example 3 – Word Permutations (apply multinomial coefficients)

How many ways to rearrange the letters in the word "GODOGGY"?

$n = 7$ (length of sequence)  $K = 4$ types $= \{G, O, D, Y\}$

$n_1 = 3$, $n_2 = 2$, $n_3 = 1$, $n_4 = 1$

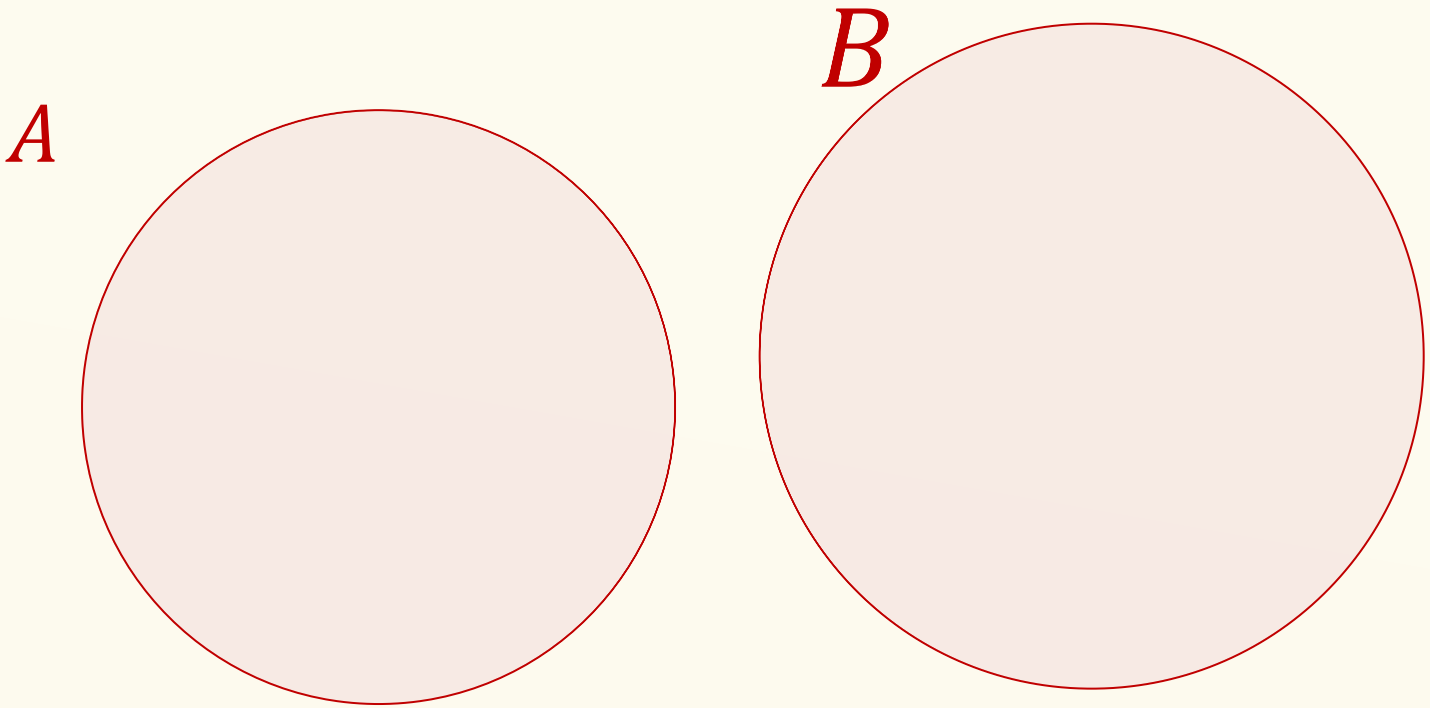
$$\binom{7}{3,2,1,1} = \frac{7!}{3!2!1!1!}$$

# Agenda (part 4)

- Recap & Examples
- Binomial Theorem
- Multinomial Coefficients
- **Inclusion-Exclusion**
- Combinatorial Proofs

# Recap Disjoint Sets

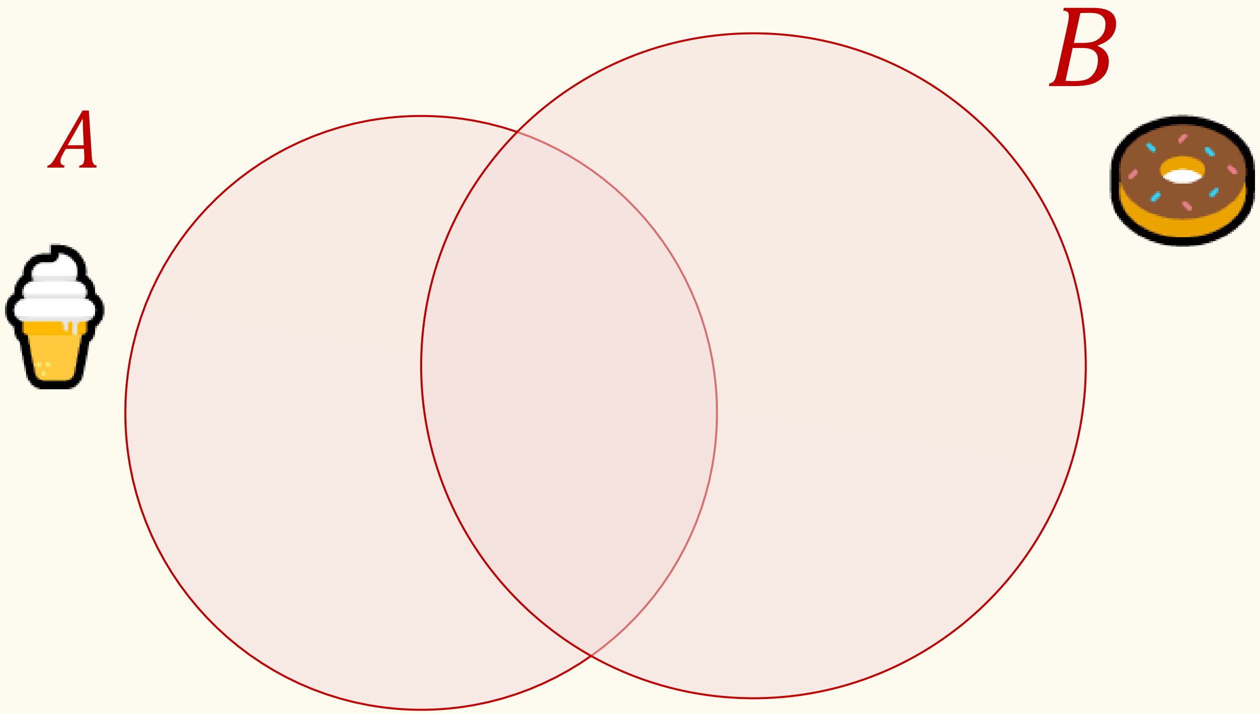
Sets that do not contain common elements ($A \cap B = \emptyset$)




**Sum Rule:** $|A \cup B| = |A| + |B|$

# Inclusion-Exclusion

But what if the sets are not disjoint?



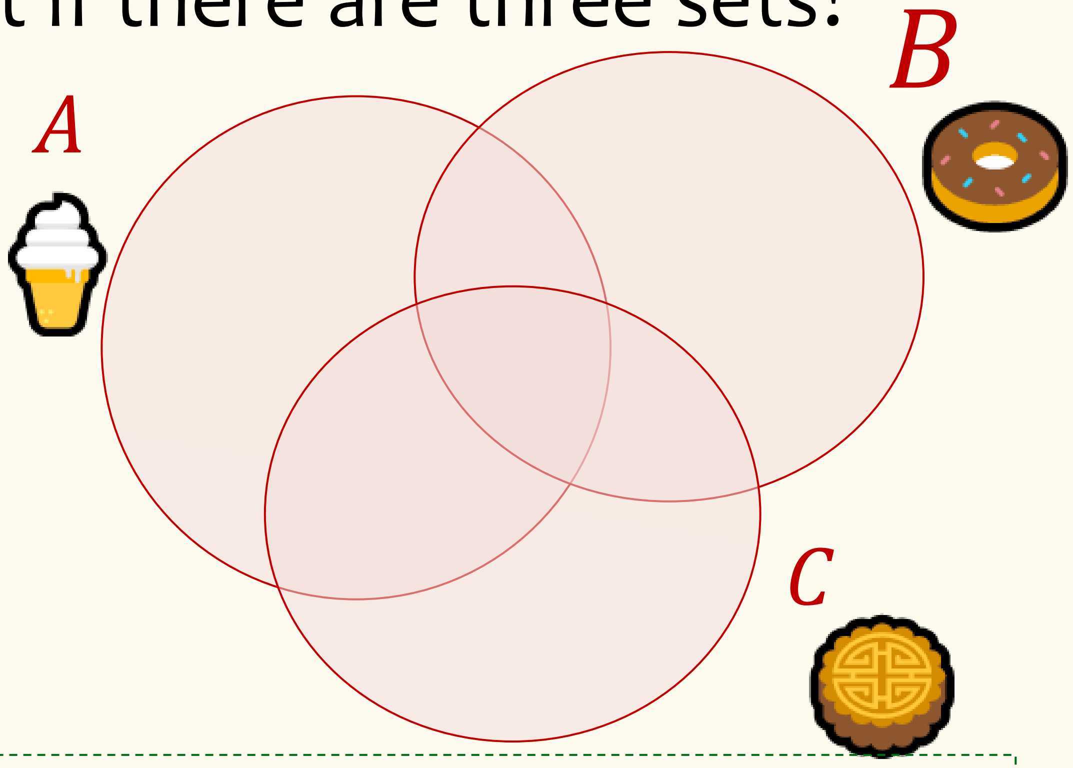

$|A| = 43$
$|B| = 20$
$|A \cap B| =$ |A intersect B| = 7
$|A \cup B| =$ |A union B| = ???

**Fact.** $|A \cup B| = |A| + |B| - |A \cap B|$

# Inclusion-Exclusion (3 sets)

Not drawn to scale

What if there are three sets?



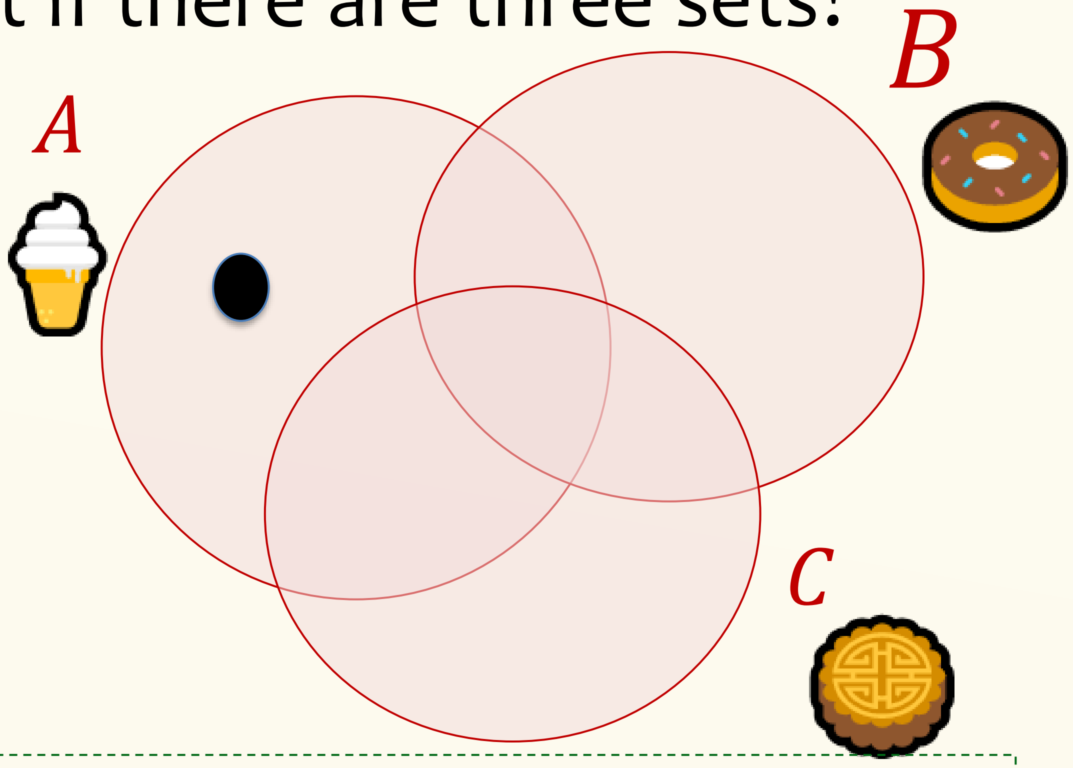

$|A| = 43$
$|B| = 20$
$|C| = 35$
$|A \cap B| = 7$
$|A \cap C| = 16$
$|B \cap C| = 11$
$|A \cap B \cap C| = 4$
$|A \cup B \cup C| = ???$

**Fact.**

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

# Inclusion-Exclusion (3 sets) – example 1

What if there are three sets?




$|A| = 43$
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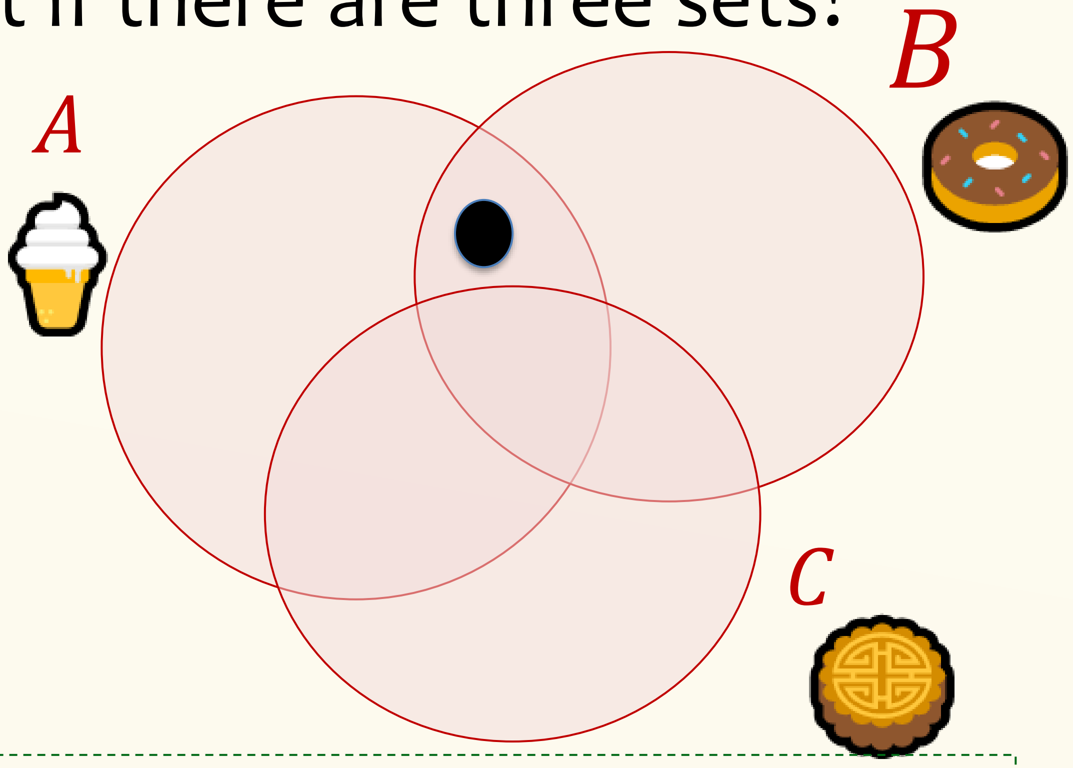
$|A \cap C| = 16$
$|B \cap C| = 11$
$|A \cap B \cap C| = 4$
$|A \cup B \cup C| = ???$

**Fact.**

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

# Inclusion-Exclusion (3 sets) – example 2

What if there are three sets?




$|A| = 43$
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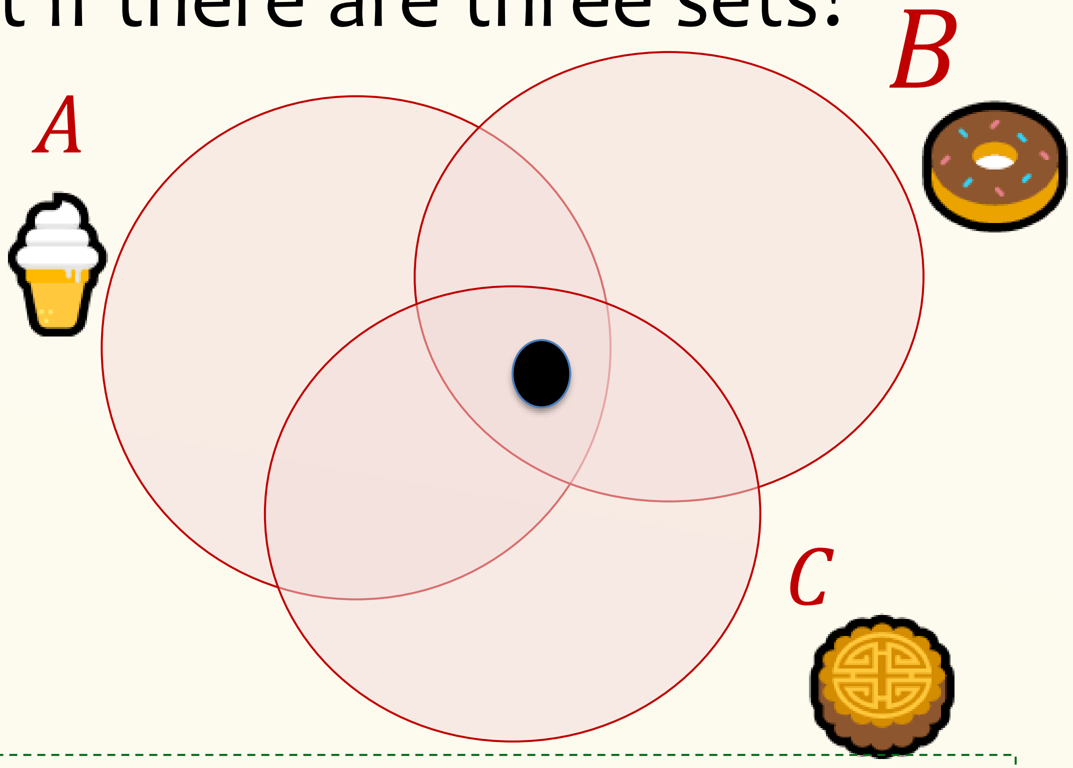
$|B \cap C| = 11$
$|A \cap B \cap C| = 4$
$|A \cup B \cup C| = ???$

**Fact.**

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

# Inclusion-Exclusion (3 sets) – example 3

What if there are three sets?




$|A| = 43$
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$|B \cap C| = 11$
$|A \cap B \cap C| = 4$
$|A \cup B \cup C| = ???$

**Fact.**

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

# Inclusion-Exclusion (formulae)

Let $A, B$ be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= singles \ - doubles + triples \ - quads + \ \ldots \\ &= (|A_1| + \cdots + |A_n|) \ - (|A_1 \cap A_2| + \ \ldots + |A_{n-1} \cap A_n|) + \ \ldots \end{aligned}$$

# Agenda (Part 5)

- Recap & Examples
- Binomial Theorem
- Multinomial Coefficients
- Inclusion-Exclusion
- Combinatorial Proofs

## Combinatorial proof: Show that $M = N$

- Let $S$ be a set of objects
- Show how to count $|S|$ one way => $|S| = M$
- Show how to count $|S|$ another way => $|S| = N$
- Conclude that $M = N$

# Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

**Fact 1.** $\binom{n}{k} = \binom{n}{n-k}$ **Symmetry in Binomial Coefficients**

**Fact 2.** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ **Pascal's Identity**

**Fact 3.** $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ **Follows from Binomial theorem**

# Pascal's Identities (Fact 2)

**Fact.** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ \ldots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

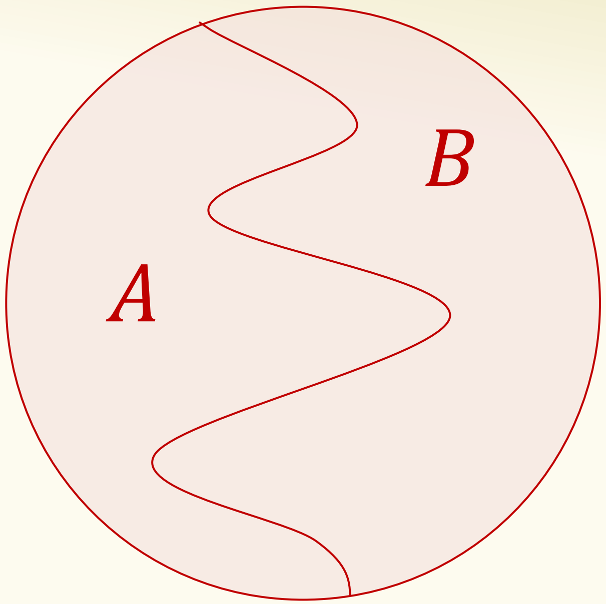
Hard work and not intuitive

Let's see a combinatorial argument

# Fact 2 – Binomial Identity

**Fact.** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \quad = \quad |A| \quad + \quad |B|$




$S = A \cup B$, disjoint

**S:** the set of size k subsets of [n] = {1, 2,., n} ➔ $|S| = \binom{n}{k}$

*A*: the set of size *k* subsets of [*n*] that contain n.

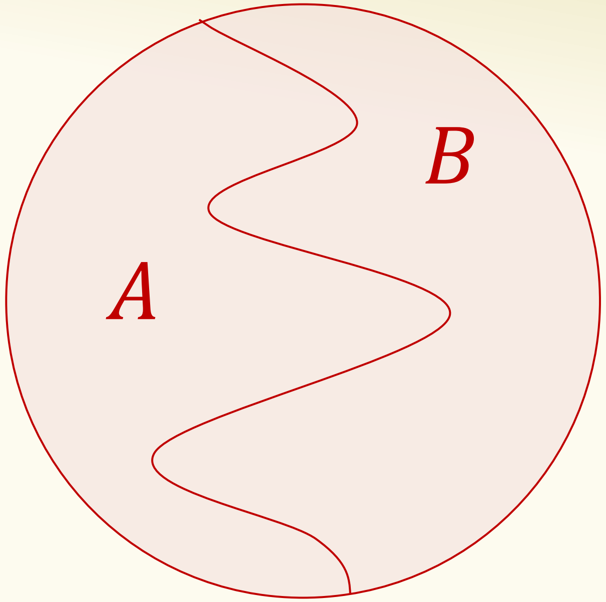
*B*: the set of size *k* subsets of [n] that do NOT contain n.

**Sum rule:**
$|A \cup B| = |A| + |B|$

# Fact 2 – Binomial Identity - example

**Fact.** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| = |A| + |B|$




$S = A \cup B$, disjoint

**S:** the set of size k subsets of [n] = {1, 2,., n} ➔ $|S| = \binom{n}{k}$

e.g.: $n = 4, \mathrm{k} = 2, \quad S = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$

*A*: the set of size *k* subsets of [*n*] that contain n.

$A = \{\{1,4\},\{2,4\},\{3,4\}\}. \qquad n = 4$

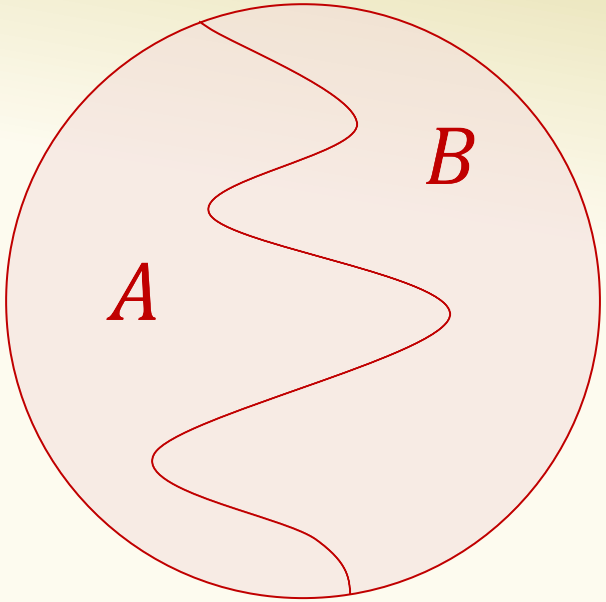
*B*: the set of size *k* subsets of [n] that do NOT contain n.

$B = \{\{1,2\},\{1,3\},\{2,3\}\}$

# Fact 2 – Binomial Identity – sizes of subsets

**Fact.** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| = |A| + |B|$




$S = A \cup B$ disjoint

$n$ is in set, need to choose $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

S: the set of size k subsets of [n] = {1, 2,., n} ➔ |S

A: the set of size $k$ subsets of $[n]$ that contain n.

B: the set of size $k$ subsets of [n] that do NOT conta

$n$ not in set, need to choose $k$ elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

# Pascal’s triangle

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$n$ =0 $\binom{0}{0}$

$n$ =1 $\binom{1}{0}$ $\binom{1}{1}$

$n$ =2 $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$n$ =3 $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$n$ =4 $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

$n$ =5 $\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

# Summary: combinatorial vs algebraic arguments

## combinatorial argument/proof

- Elegant
- Simple
- Intuitive

## Algebraic argument

- Brute force
- Less Intuitive