Normal distribution + Central Limit Theorem

CSE 312 Spring 26

Lecture 17

Recap – Continuous RVs

Probability Density Function (PDF).

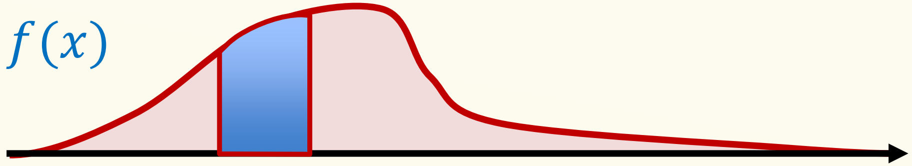
$f: \mathbb{R} \to \mathbb{R}$ s.t.

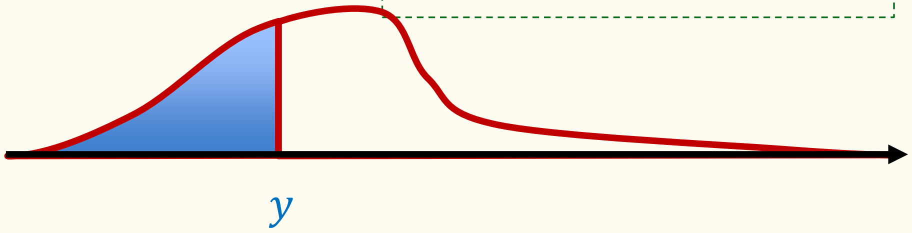
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = 1$

Cumulative Distribution Function (CDF).

$$F(y) = \int_{-\infty}^{y} f(x)\, \mathrm{d}x$$

Theorem. $f(x) = \frac{dF(x)}{dx}$





Density ≠ Probability !

$$P(X \in [a, b]) = \int_a^b f_X(x) \mathrm{d}x$$
$$= F_X(b) - F_X(a)$$

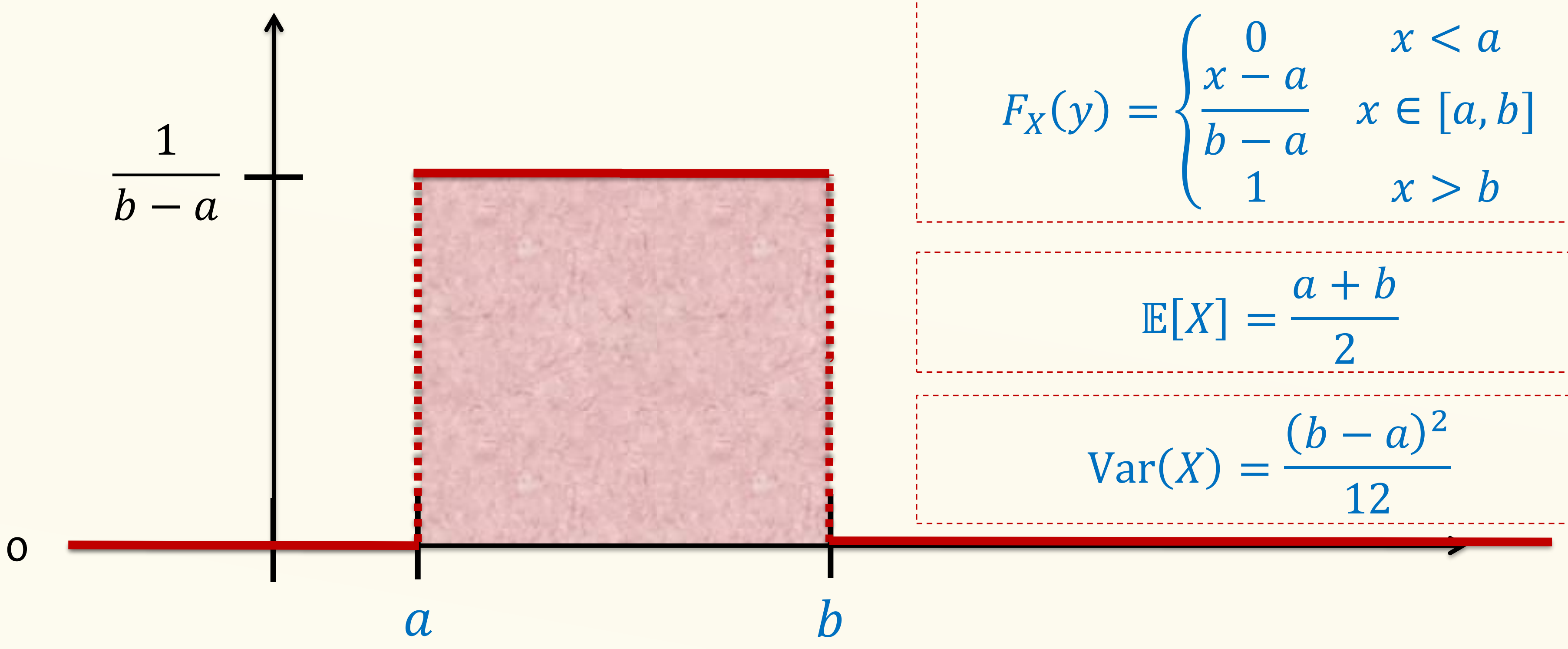
$$F_X(y) = P(X \leq y)$$

Recap: From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_{x} p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_{x} g(x)\, p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$

Uniform Distribution Summary

$X \sim \text{Unif}(a, b)$



$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

$P(X > t) = e^{-t\lambda}$

Exponential Distribution

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

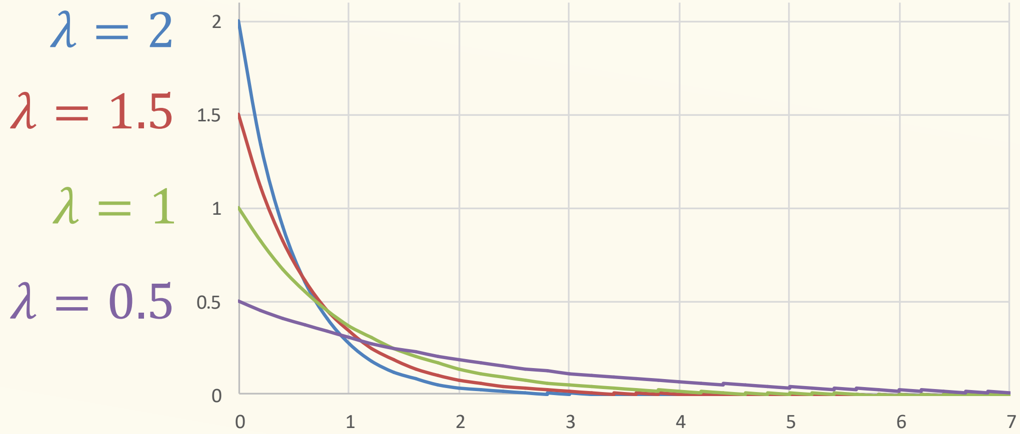
Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$

$\mathbb{E}[X] = \frac{1}{\lambda}$

$\mathrm{Var}(X) = \frac{1}{\lambda^2}$



Agenda

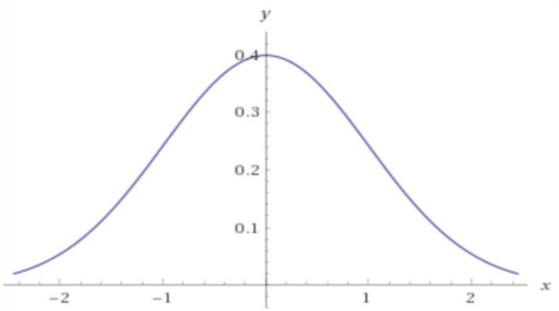
- Normal Distribution Practice with Normals
- Central Limit Theorem (CLT)

The Normal Distribution

Definition. A **Gaussian (or <u>normal</u>) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.



$\mathcal{N}(0,1)$.

Carl Friedrich Gauss

The Normal Distribution

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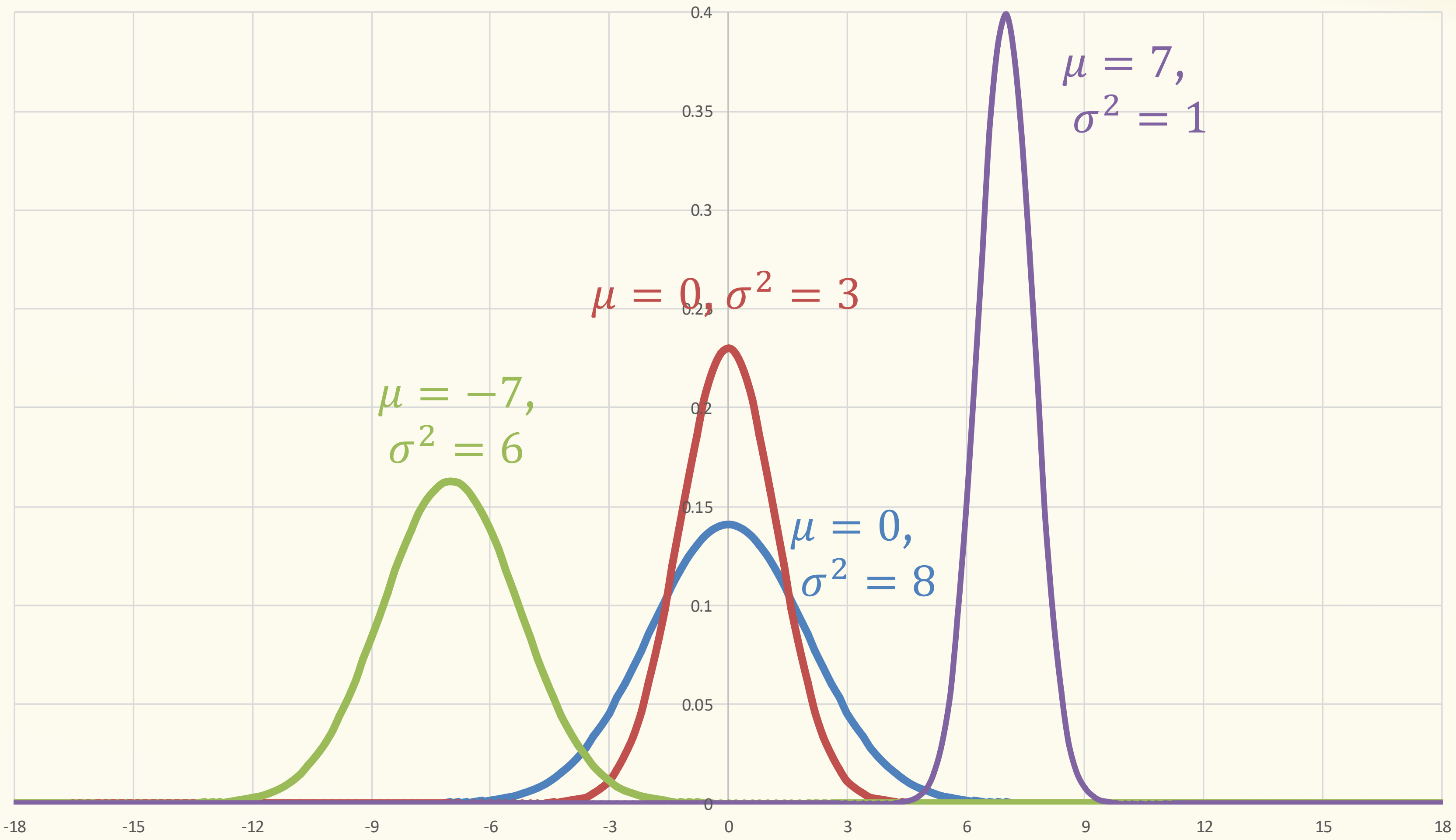
We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Carl Friedrich Gauss

The Normal Distribution

Aka a "Bell Curve" (imprecise name)



Standard normal distribution

Z

Standard (unit) normal = $\mathcal{N}(0,1)$

CDF. $\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0,1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

Review Table of $\Phi(z)$ CDF of Standard Normal

$P(Z \leq 0.98) = \Phi(0.98) \approx 0.8365$

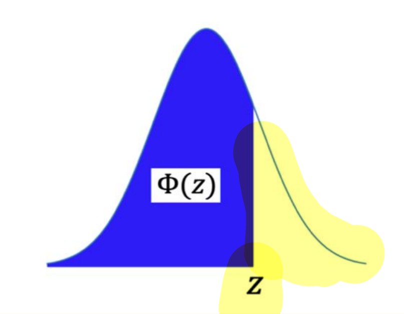
For what a is

$P(Z > a) \leq 0.01$?

$P(Z \leq a) = 1 - 0.01 = 0.99$

$P(Z \leq 2.33) = 0.9901$

$> \quad \geq \quad < \quad <$



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

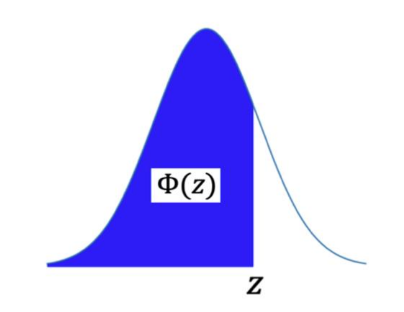
Review Table of $\Phi(z)$ CDF of Standard Normal

$P(Z \leq 0.98) = \Phi(0.98) \approx 0.8365$

For what a is

$P(Z \geq a) \leq 0.01$?

For any $a \geq 2.33$
$P(Z > a) \leq 0.01.$



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3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

The Standard Normal CDF

$Z \sim N(0,1)$

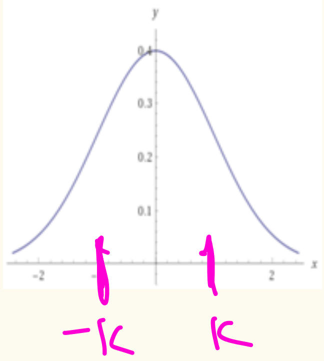
$\sigma^2 = 1$ $\sigma = 1$

What is the probability that a standard Normal is within k standard deviations of its mean?

$$P(-k < Z < k) = \Phi(k) - \Phi(-k)$$

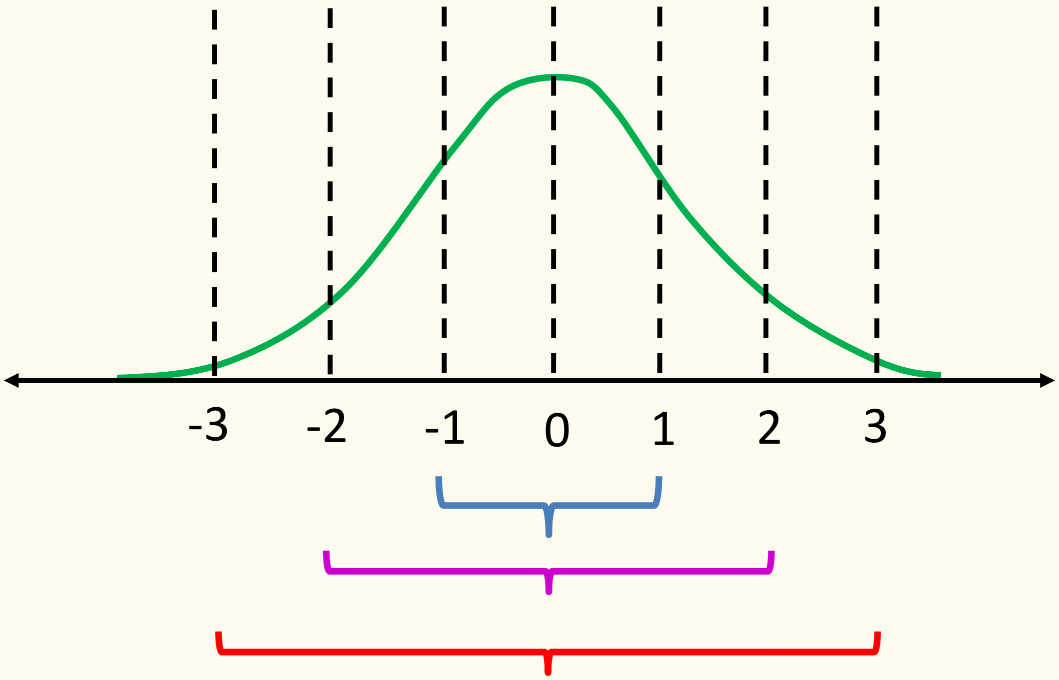
$$= \Phi(k) - (1 - \Phi(k))$$

$$= 2 \cdot \Phi(k) - 1$$



Deviation from the Mean

If $Z \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbf{P}\{-k < Z < k\} = 2\boldsymbol{\Phi}(k) - 1$



- w/prob 68%, Z is within 1 std of its mean
- w/prob 95%, Z is within 2 std of its mean
- w/prob 99.7%, Z is within 3 std of its mean

Closure of normal distribution – Under Shifting and Scaling

standardized version of X

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Mean and variance follow from properties you know! The fact that result of shifting and scaling still normal is not obvious, but not too difficult

X r.v.

$E(X) = \mu$

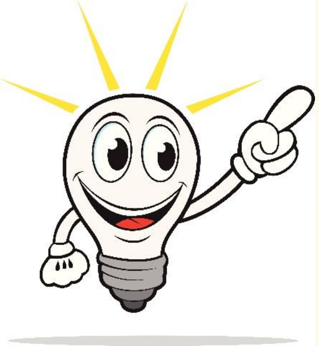
$Var(X) = \sigma^2$

$\frac{X-\mu}{\sigma}$ has exp value 0

variance 1

But what if we don't have a standard Normal?

<u>Bottom line</u>: Deviations from mean we saw for a standard Normal holds for general Normal (provided it's phrased in terms of standard deviations).



$$X \sim \mathcal{N}(\mu, \sigma^2) \quad \Longleftrightarrow \quad Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

$$\boldsymbol{P}\{-k\sigma < X - \mu < k\sigma\} = \boldsymbol{P}\left\{-k < \frac{X - \mu}{\sigma} < k\right\} = \boldsymbol{P}\{-k < Z < k\}$$

Prob. X deviates from its mean by k stds

Prob. Z deviates from its mean by k stds

Agenda

- Normal Distribution
- Practice with Normals
- The Central Limit Theorem

Example

$\sigma^2 = 4 \Rightarrow \sigma = 2$

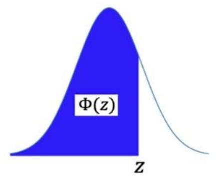
Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$P(X \leq 1.2) = P\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right)$

$\frac{X-0.4}{2} \sim N(0,1)$, $\quad \frac{1.2-0.4}{2} = 0.4$

$= P(Z \leq 0.4) = 0.65542$

Table of Standard Cumulative Normal Density



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1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
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2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Example

Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$$P(X \leq 1.2) = P\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right)$$

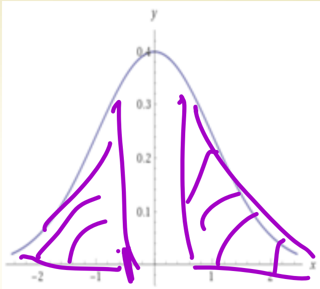
$$= P\left(\frac{X - 0.4}{2} \leq 0.4\right) = \Phi(0.4) \approx 0.6554$$

$\sim \mathcal{N}(0, 1)$

0.1	0.5398	0.5438
0.2	0.5793	0.5832
0.3	0.6179	0.6217
0.4	0.6554	0.6591
0.5	0.6915	0.6950
0.6	0.7257	0.7291
0.7	0.7580	0.7611

Example

$\sigma = 4$



Let $X \sim \mathcal{N}(3, 16)$.

$$P(2 < X < 5) = P\left(\frac{2-3}{4} < \frac{X-3}{4} < \frac{5-3}{4}\right)$$

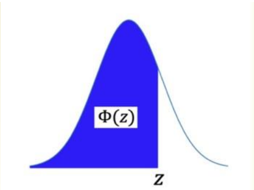
$$= P\left(-\frac{1}{4} < Z < \frac{1}{2}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right)$$

$$1 - \Phi\left(\frac{1}{4}\right)$$

$$= \Phi\left(\frac{1}{2}\right) + \Phi\left(\frac{1}{4}\right) - 1$$

Table of Standard Cumulative Normal Density



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
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0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
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3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Example

Let $X \sim \mathcal{N}(3, 16)$.

$$
\begin{aligned}
P(2 < X < 5) &= P\left(\frac{2 - 3}{4} < \frac{X - 3}{4} < \frac{5 - 3}{4}\right) \\
&= P\left(-\frac{1}{4} < Z < \frac{1}{2}\right) \\
&= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right) \\
&= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right) \approx 0.29017
\end{aligned}
$$

Summary so far

- Normal distributions stay normal under shifting and scaling.

- To "standardize" a normal random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, you subtract the mean and divide by the standard deviation, i.e.,

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$

- This allows you to use the standard normal tables (showing $\Phi(z) = P(Z \le z)$ for $Z \sim \mathcal{N}(0, 1)$) to do calculations for any normal distribution.

Another important property: closure under addition

Fact. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $\mathrm{Y} \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV) then $\mathrm{a}X + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that **the sum of normal RVs is still a normal RV**. The values of the expectation and variance are **not** surprising.

Why not surprising?

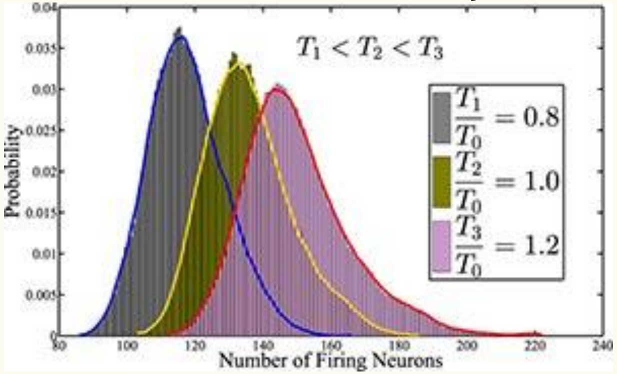
- Linearity of expectation (always true)
- When X and Y are independent, $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

Agenda

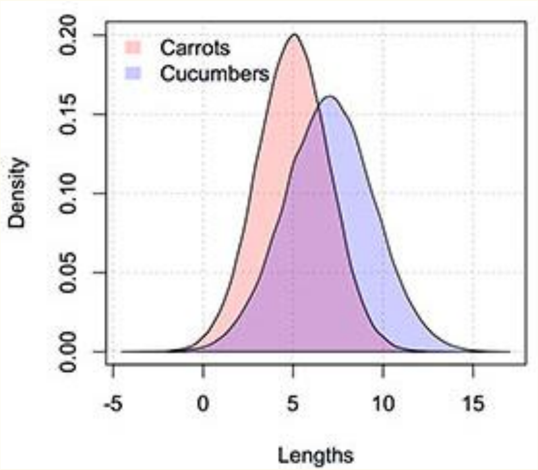
- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT)

Normal Distributions EVERYWHERE – why?

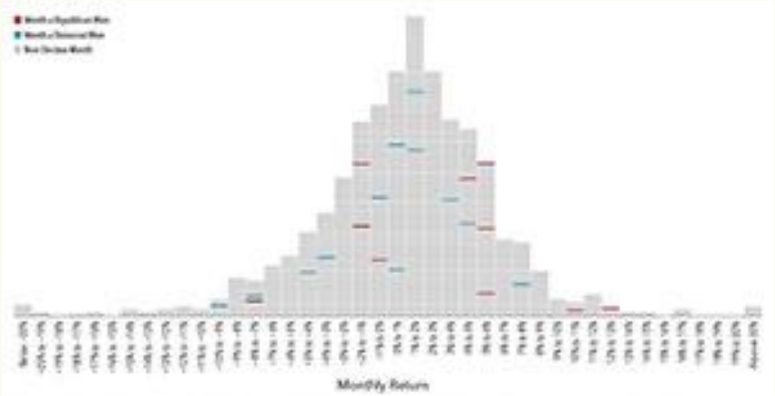
Neuron Activity



Vegetables



S&P 500 Returns after Elections



Examples from:
https://galtonboard.com/probabilityexamplesinlife

Sums of i.i.d. RVs look normal!

i.i.d. = independent and identically distributed

$X_1, \dots, X_n$ i.i.d. with expectation μ and variance σ^2

Consider $S_n = X_1 + \cdots + X_n$

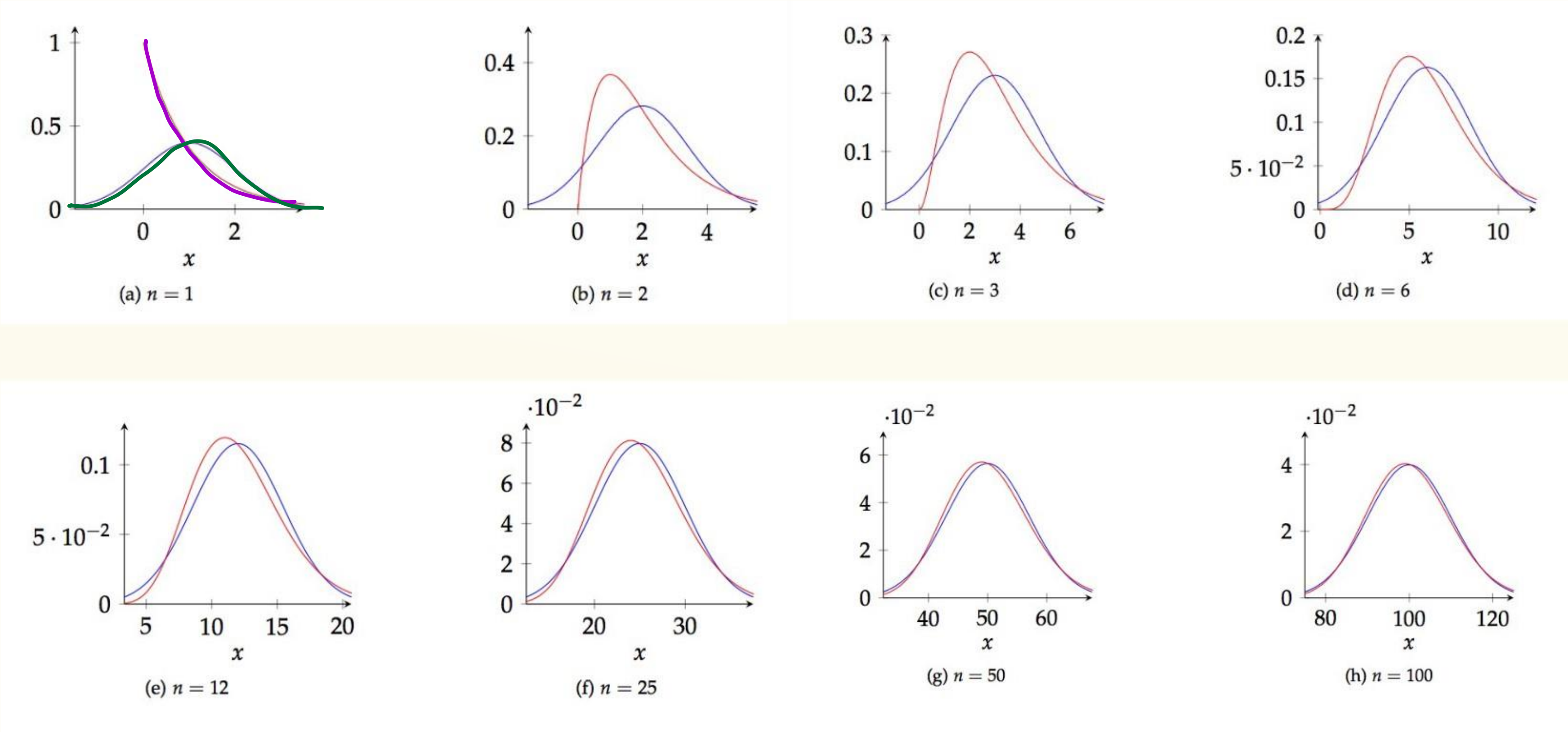
$\mathbb{E}[S_n] =$ $n\mu$

$\text{Var}(S_n) =$ $n\sigma^2.$

Empirical observation:

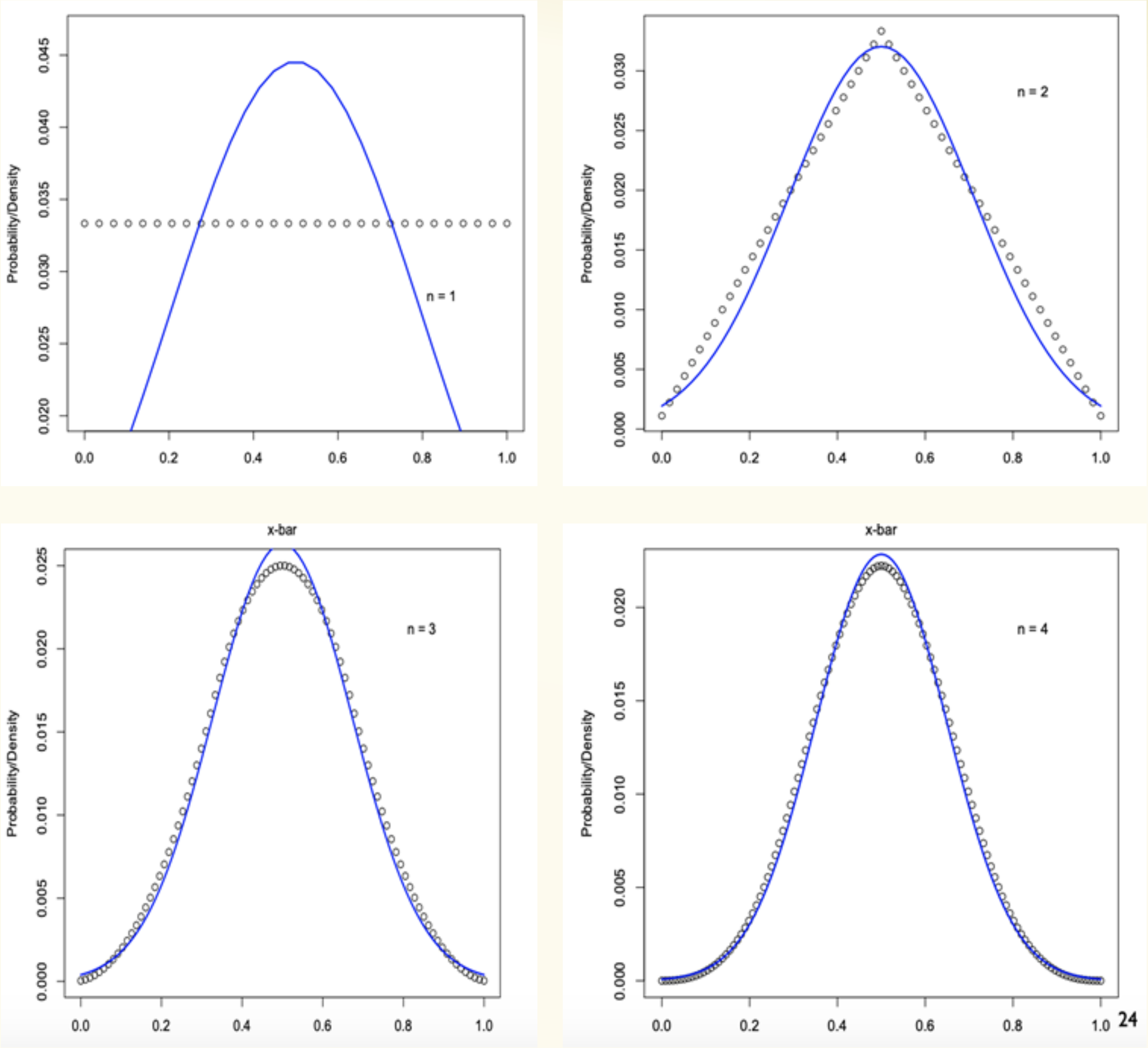
S_n looks like a normal RV as n grows.

Example: Sum of n i.i.d. $\mathrm{Exp}(1)$ random variables

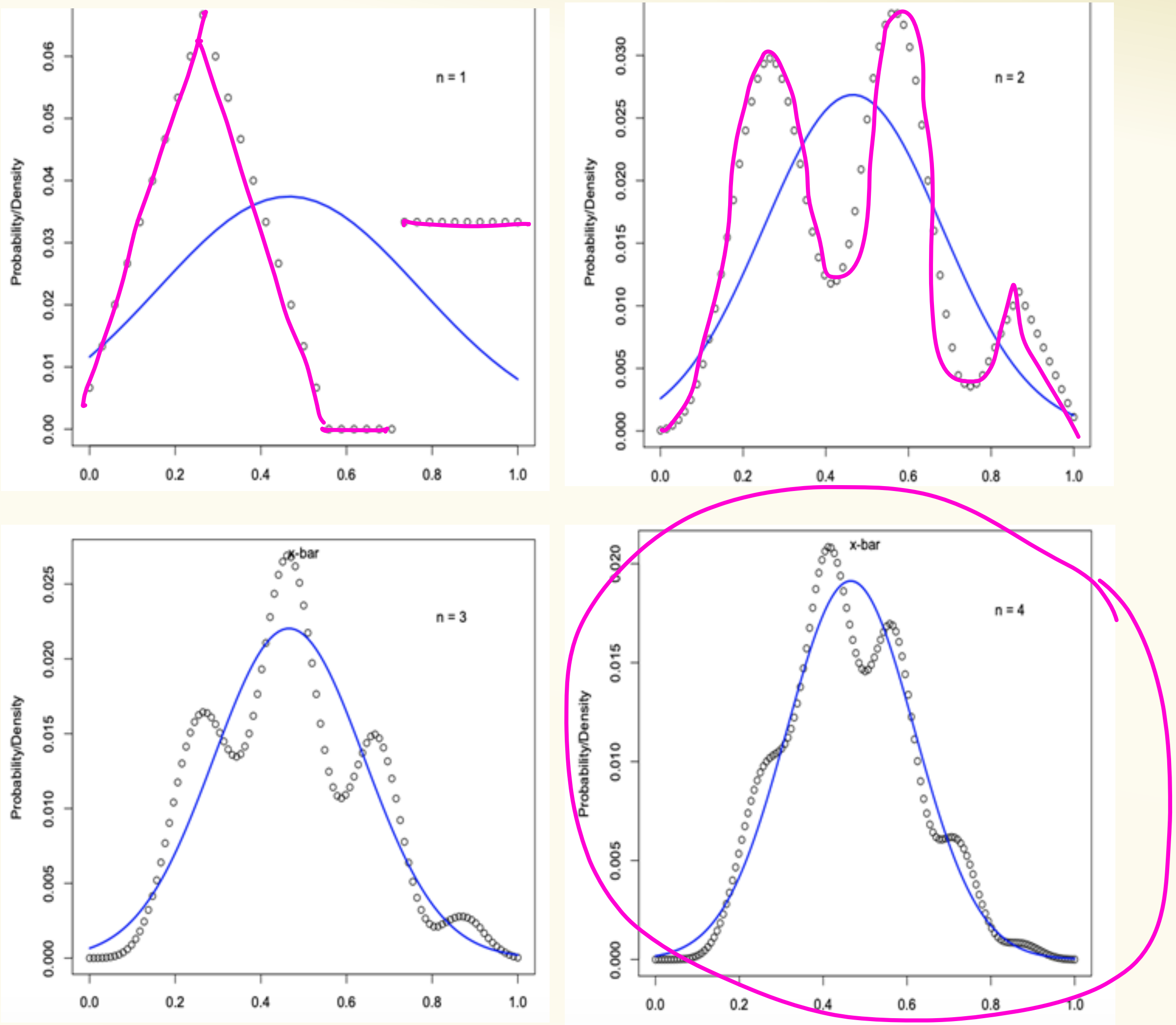


Example: avg of uniform r.v.s

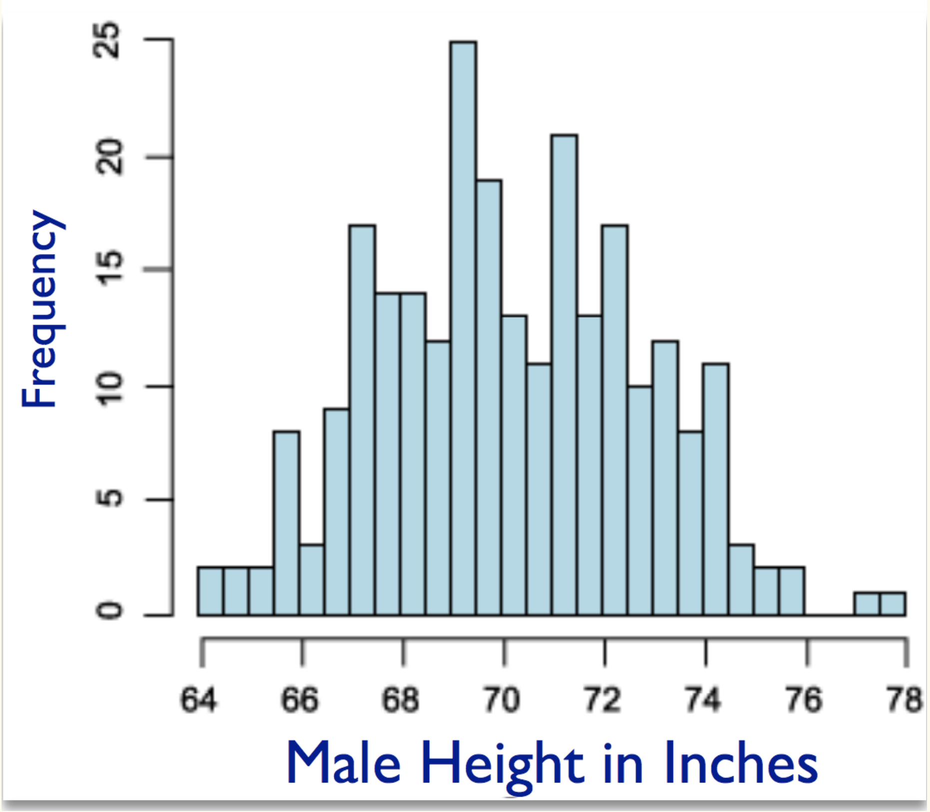
$$\frac{X_1 + X_2 + \dots + X_n}{n}$$



CLT : Avg of some other weird i.i.d. r.v.s



Suppose that what we see in nature results from combining (summing) many independent random observations…



Then distribution might look normal. e.g. Height distribution resembles Gaussian.

R.A.Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can written as

$$X = X_1 + \cdots + X_n$$

Sums of i.i.d. RVs

i.i.d. = independent and identically distributed

$X_1, \dots, X_n$ i.i.d. with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$

$$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n\mu$$

$$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$$

Empirical observation:

S_n looks like a normal RV as n grows, but…

Central Limit Theorem

$X_1, \ldots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n\mu$

$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$

$\sigma(S_n) = \sqrt{n}\,\sigma$

$\mathbb{E}[Y_n] = 0$

$\mathrm{Var}(Y_n) = 1$

Central Limit Theorem

$X_1, \ldots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and

$$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] \quad = n\mu$$

$$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) \quad = n\sigma^2$$

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}[Y_n] = \frac{1}{\sigma\sqrt{n}}(\mathbb{E}[S_n] - n\mu) = \frac{1}{\sigma\sqrt{n}}(n\mu - n\mu) = 0$$

$$\mathrm{Var}(Y_n) = \frac{1}{\sigma^2 n}\left(\mathrm{Var}(S_n - n\mu)\right) = \frac{\mathrm{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

Central Limit Theorem

$X_1, \dots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$,

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

Then distribution of $Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$ converges to that of a normal distribution with mean 0 and variance 1 as $n \to \infty$.

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n\to\infty} P(Y_n \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

$= \Phi(y)$

$= P(Z \le y)$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

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$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\mathbb{E}\left(\sum X_i\right) = \frac{1}{n}\sum \mathbb{E}(X_i) = \frac{1}{n}\cdot n\mu = \mu$$

Also stated as:

- $\lim_{n\to\infty} Y_n \to \mathcal{N}(0,1)$
- $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$ for $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = \mathrm{Var}(X_i)$

Summary Central Limit Theorem

$$Var\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2} Var\left(\sum X_i\right) = \frac{1}{n^2}\cdot n\sigma^2 = \sigma^2/n$$

$X_1, \ldots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. and $Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$

	S_n	$\bar{X} = \frac{1}{n}\sum X_i$	Y_n
mean	$n\mu$	μ	0
variance	$n\sigma^2$	$\frac{\sigma^2}{n}$	1
CLT:	$\approx N(n\mu, n\sigma^2)$	$\xrightarrow{n\to\infty} N\left(\mu, \frac{\sigma^2}{n}\right)$	$\xrightarrow{n\to\infty} N(0,1)$

CLT application

- Suppose lightbulbs have a lifetime that is exponential with a mean of 5 hours.
- You buy a pack of 10 light bulbs. As soon as one burns out you replace it with another. What is the probability you still have a working light bulb after 70 hours?
- Estimate using the Central Limit Theorem.

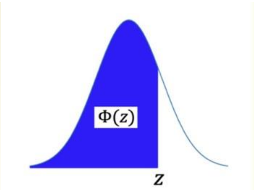
X_i: time for i^{th} lightbulb to burn out. $E(X_i) = \boxed{5} = \frac{1}{\lambda}$ $\quad X_i \sim Exp(\frac{1}{5})$

$Var(X_i) = 25$

$X = X_1 + X_2 + \dots + X_{10} \qquad X \approx N(50, 250)$

$$P(X > 70) = P\left(\underbrace{\frac{X-50}{\sqrt{250}}}_{N(0,1)} > \frac{70-50}{\sqrt{250}}\right)$$

Table of Standard Cumulative Normal Density



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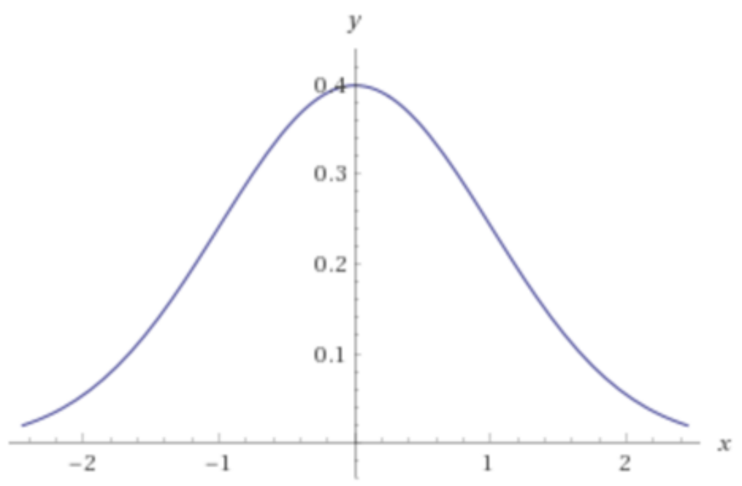
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Outline of how CLT is used

- Write the event you are interested in, in terms of a sum of i.i.d. random variables.
- Normalize RV to have mean 0 and standard deviation 1.
- Write event in terms of Φ, the CDF of a $\mathcal{N}(0,1)$.
- Look up in table.



Normal Distribution

Paranormal Distribution