Continuous Random Variables: Exponential and Normal

CSE 312 Spring 26

Lecture 16

Some slides from book by Harchol-Balter.

Definition. A **continuous random variable** X has a continuous range of values that it can take (an interval or a set of intervals).

Thus, a continuous random variable can take on an uncountable set of possible values

Examples:

- ❑ Time of an event
- ❑ Response time of a job
- ❑ Speed of a device
- ❑ Location of a satellite
- ❑ Distance between people’s eyeballs

Recap – Continuous RVs

$P(X \leq x) = F_X(x)$

Probability Density Function (PDF).

$f: \mathbb{R} \to \mathbb{R}$ s.t.

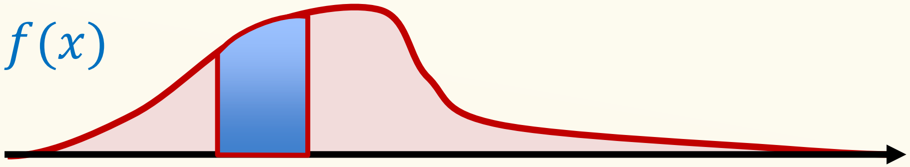
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = 1$

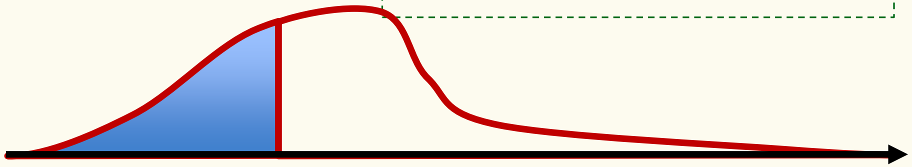
Cumulative Distribution Function (CDF).

$$F(y) = \int_{-\infty}^{y} f(x)\, \mathrm{d}x$$

Theorem. $f(x) = \dfrac{dF(x)}{dx}$

$f(x)$





y

Density ≠ Probability !

$$P(X \in [a,b]) = \int_a^b f_X(x)\mathrm{d}x$$
$$= F_X(b) - F_X(a)$$

$$F_X(y) = P(X \leq y)$$
$$= P(X < y)$$

Expectation of a Continuous RV

$E(X) = \sum_{x \in \Omega_X} x \, p_X(x)$

Definition. The **expected value** of a continuous RV X is defined as

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Fact. $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$

Proofs follow same ideas as discrete case

Definition. The **variance** of a continuous RV X is defined as

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}[X])^2 \, \mathrm{d}x = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

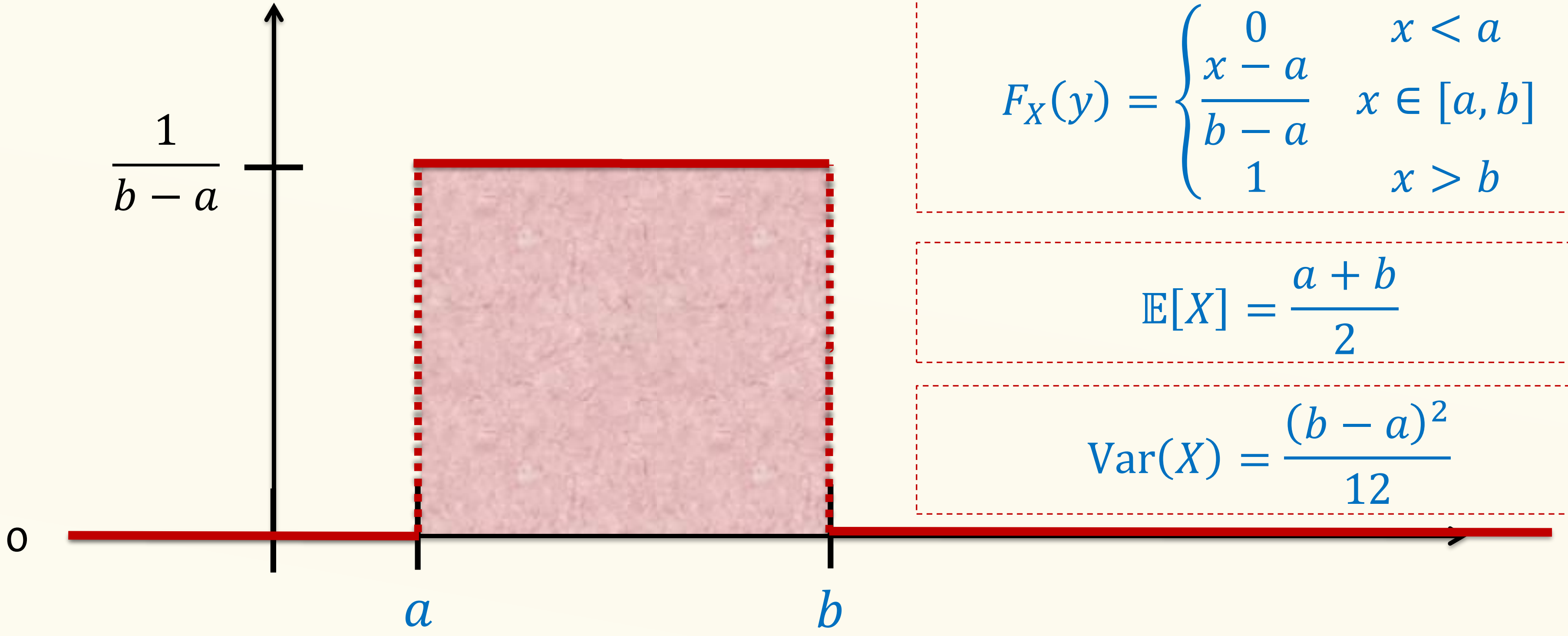
$\mathrm{Var}(X) = \sum_{x \in \Omega} (x - E[X])^2 \, p_X(x)$

Recap: From Discrete to Continuous

	Discrete	**Continuous**
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_{x} p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_{x} g(x)\, p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$

Uniform Distribution Summary

$X \sim \mathrm{Unif}(a, b)$



$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

$P(X > t) = e^{-t\lambda}$

Exponential Distribution

We write $X \sim \text{Exp}(\lambda)$ and say X that follows the exponential distribution.

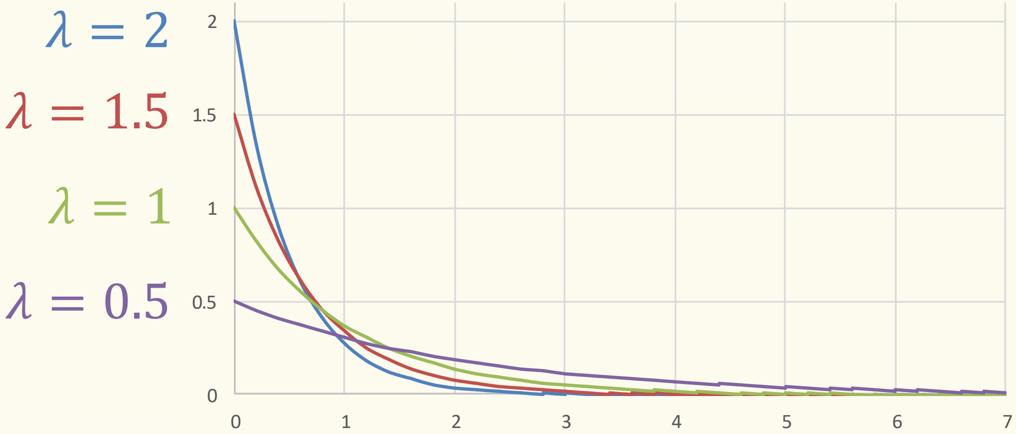
Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$



$$W \sim Poi(\lambda) \Rightarrow P(W = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is λ (independently)

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0,1,2, \ldots\}$
- Let $X \sim Exp(\lambda)$ be the time till the first event. We will compute $F_X(t)$ and $f_X(t)$
- We know $Z_t \sim Poi(t\lambda)$ be the # of events in the first t units of time, for $t \geq 0$.
- $P(X > t) = P(\text{no event in the first } t \text{ units}) = P(Z_t = 0) = e^{-t\lambda} \frac{(t\lambda)^0}{0!} = e^{-t\lambda}$
- $F_X(t) = P(X \leq t) = 1 - P(Y > t) = 1 - e^{-t\lambda}$
- $f_X(t) = \frac{d}{dt} F_X(t) = \lambda e^{-t\lambda}$

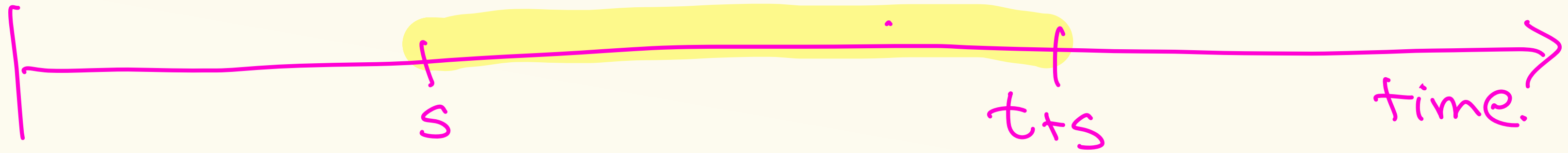
Memorylessness

Definition. A random variable is **memoryless** if for all $s, t > 0$,

$$P(X > s + t \mid X > s) = P(X > t).$$

Fact. $X \sim \mathrm{Exp}(\lambda)$ is memoryless.

Assuming an exponential distribution, if you've waited s minutes,
The probability of waiting t more is exactly same as when $s = 0$.



$$P(X > t) = e^{-\lambda t}$$

Memorylessness of Exponential

Proof that assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as when $s = 0$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$$P(X > s + t \mid X > s) = \frac{P(X>s+t,\ X>s)}{P(X>s)} = \frac{P(X>s+t)}{P(X>s)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}$$

Memorylessness of Exponential

$P(X > t) = e^{-\lambda t}$

Proof that assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as when $s = 0$

Fact. $X \sim \mathrm{Exp}(\lambda)$ is memoryless.

Proof.

$$P(X > s + t \mid X > s) = \frac{P(\{X > s + t\} \cap \{X > s\})}{P(X > s)}$$

$$= \frac{P(X > s + t)}{P(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

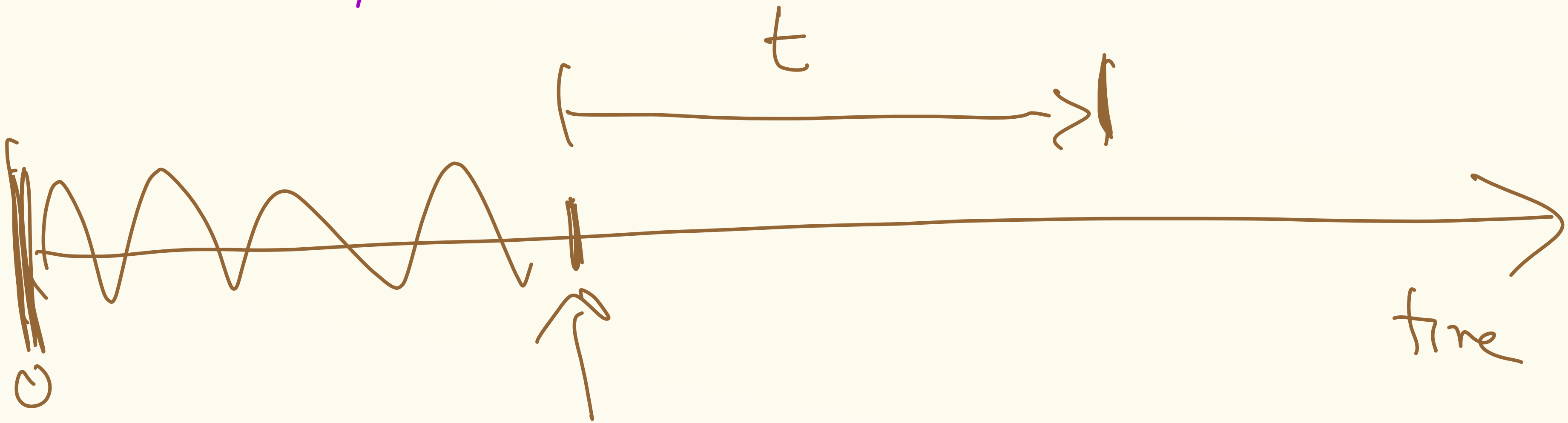
The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous)

$Y \sim Geo(p)$ $\quad P(Y > k) = (1-p)^k$

$$P(Y > k+r \mid Y > k) = \frac{P(Y > k+r,\ Y > k)}{P(Y > k)} = \frac{P(Y > k+r)}{P(Y > k)}$$

$$= \frac{(1-p)^{k+r}}{(1-p)^k} = (1-p)^r$$

$$= P(Y > r)$$



Example

T

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If there is one person currently being served, what is the probability that you will have to wait between 10 and 20 mins until you start getting service?

$$F_T(t) = 1 - e^{-\lambda t}$$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

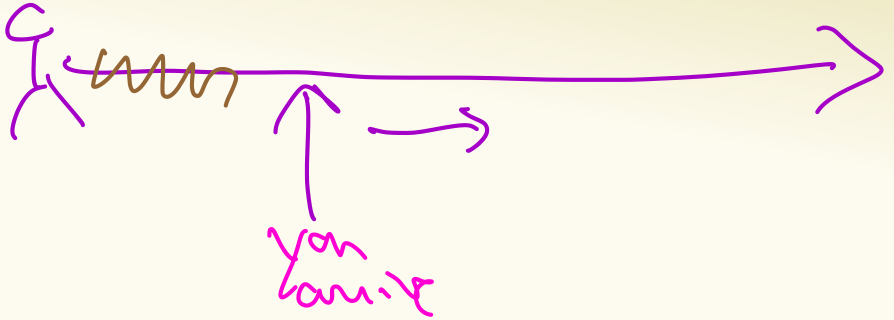
$$E(T) = \frac{1}{\lambda} = 10 \text{ mins}$$

$$\lambda = \frac{1}{10}$$

exp distn memoryless

amt of time you have to wait

you $\sim \mathrm{Exp}\left(\frac{1}{10}\right)$



$$P(10 \le T \le 20) = \int_{10}^{20} f_T(t)\,dt = \int_{10}^{20} \frac{1}{10} e^{-t/10}\,dt$$

$x = \frac{t}{10}$

$dx = \frac{dt}{10}$

$$\int_1^2 e^{-x}\,dx = e^{-1} - e^{-2}.$$

$$= F_T(20) - F_T(10) = \underbrace{1 - e^{-\frac{20}{10}}} - \left(1 - e^{-\frac{10}{10}}\right)$$

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If there is one person currently being served, what is the probability that you will have to wait between 10 and 20 mins until you start getting service?

$$T \sim Exp(\frac{1}{10})$$

$$P(10 \le T \le 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}}\, dx$$

$y = \frac{x}{10}$ so $dy = \frac{dx}{10}$

$$P(10 \le T \le 20) = \int_{1}^{2} e^{-y}\, dy = -e^{-y}\Big|_1^2 = e^{-1} - e^{-2}$$

Agenda

- Zoo
  - Exponential Distribution
  - Normal Distribution

The Normal Distribution

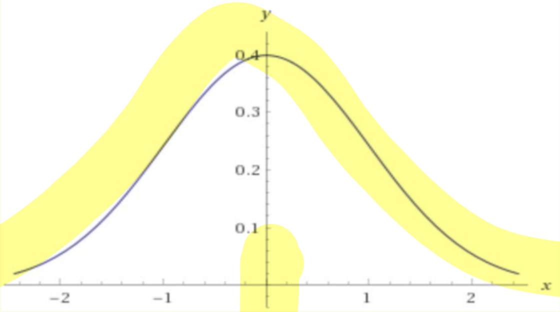
Definition. A **Gaussian (or <u>normal</u>) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Carl Friedrich Gauss

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$



$\mathcal{N}(0,1)$.

The Normal Distribution

Carl Friedrich Gauss

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We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$, and $\mathrm{Var}(X) = \sigma^2$

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

The Normal Distribution

Carl Friedrich Gauss

Definition. A **Gaussian (or normal) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$\mu+a$
$\mu-a$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.

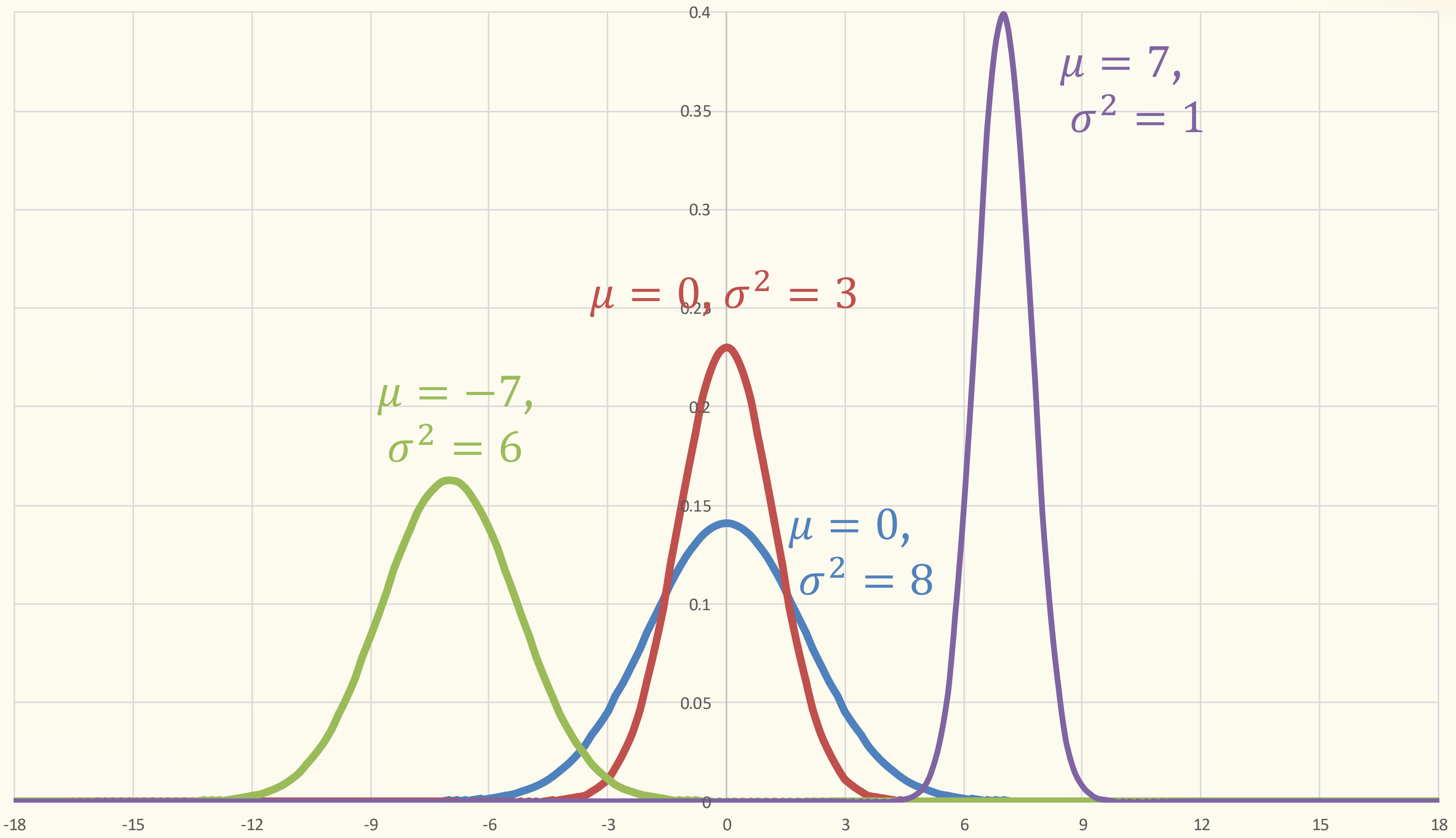
Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Proof of expectation is easy because density curve is symmetric around μ, $f_X(\mu - x) = f_X(\mu + x)$, but proof for variance requires integration of $e^{-x^2/2}$

We will see next time why the normal distribution is (in some sense) the most important distribution.

The Normal Distribution

Aka a "Bell Curve" (imprecise name)



Standard normal distribution

Standard (unit) normal = $\mathcal{N}(0,1)$

CDF. $\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0,1)$

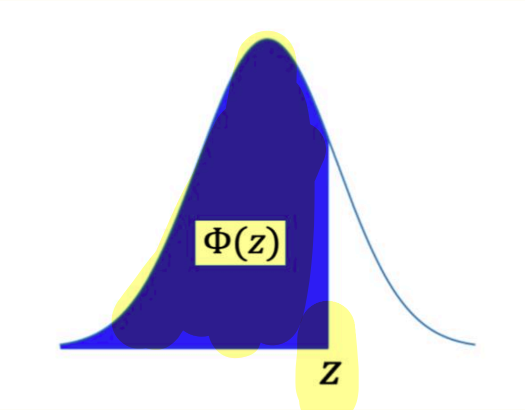
Note: $\Phi(z)$ has no closed form – generally given via tables

Table of Standard Cumulative Normal Density $\mathcal{N}(0,1)$

$P(Z < 1.35)$

$P(Z \leq 0.98) = \Phi(0.98) \approx 0.8365$

$P(Z \leq 1) = \Phi(1.00) \approx 0.84134$



1.35

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

$P(Z \leq 1) \;=\Phi(1.00)\;\approx 0.84$

The Standard Normal CDF

What is the probability that a standard Normal is within one standard deviation of its mean?

$\Phi(-1.3) = 1 - \Phi(1.3)$

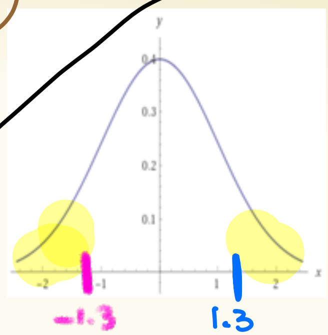
$P(-1 \leq Z \leq 1)=$

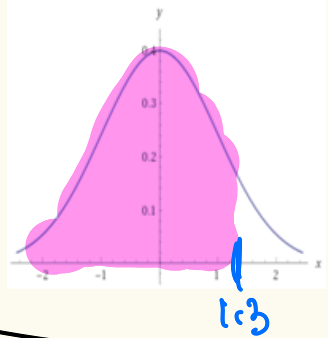
$= \Phi(1) - \overset{P(Z<-1)}{\Phi(-1)}$

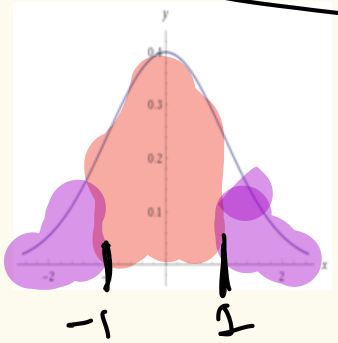
$= \Phi(1) - \underbrace{P(Z>1)}_{=1-P(Z\leq 1)}$

$= \Phi(1) - (1-\Phi(1)) = 2\Phi(1) - 1 \approx 0.68$

$P(-k \leq Z \leq k) = 2\Phi(k) - 1$

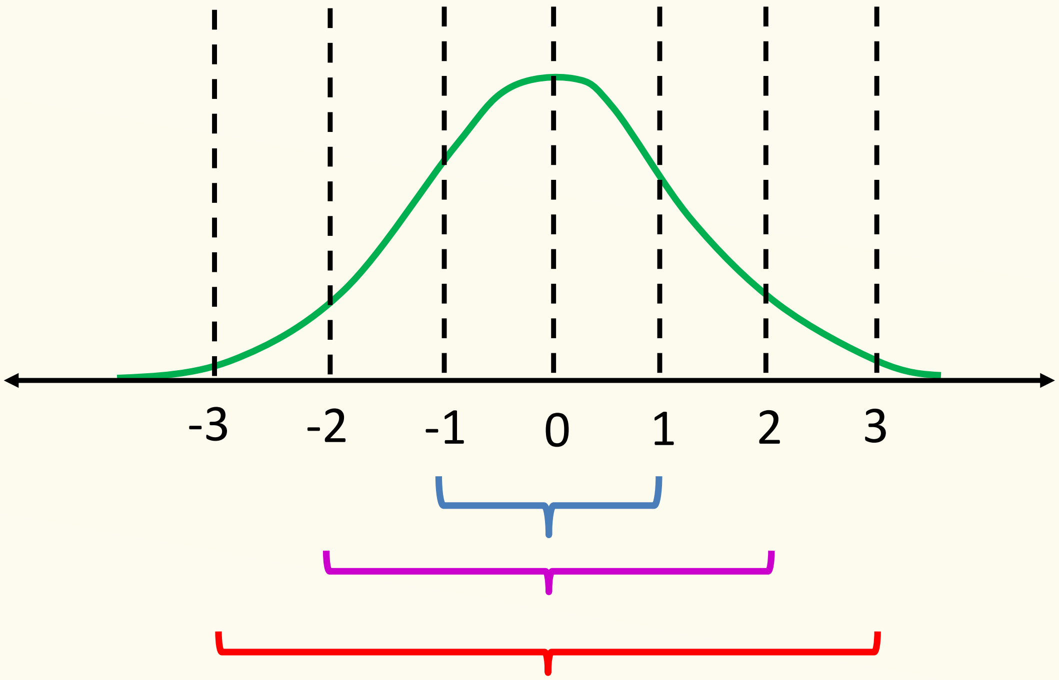






Deviation from the Mean

If $Z \sim Normal(0, 1)$, then $\mathbf{P}\{-k < Z < k\} = 2\mathbf{\Phi}(k) - 1$



- w/prob 68%, Z is within 1 std of its mean
- w/prob 95%, Z is within 2 std of its mean
- w/prob 99.7%, Z is within 3 std of its mean

Closure of normal distribution – Under Shifting and Scaling

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Mean and variance follow from properties you know! The fact that $\frac{X-\mu}{\sigma}$ is still normal is not obvious, but not too difficult

X r.v. $\quad Y = \frac{X-\mu}{\sigma} \Longleftarrow$ standardizing a random variable

$$E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma} E(X-\mu) = \frac{1}{\sigma}\left(\underbrace{E(X)}_{\mu} - \mu\right) = 0$$

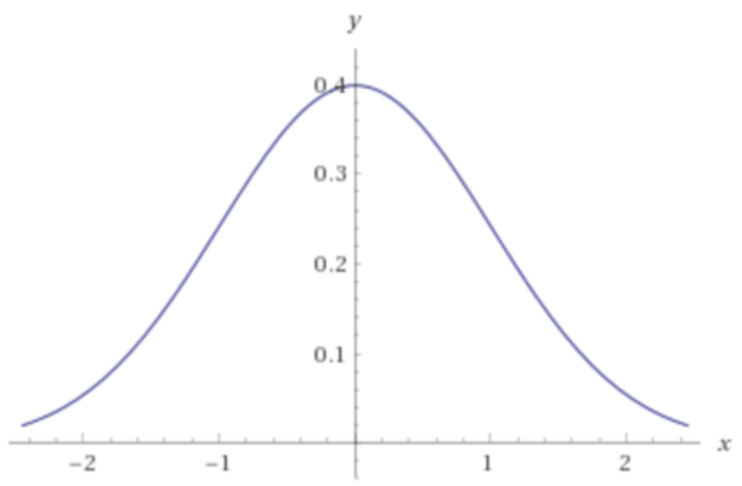
$$Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X-\mu) = \frac{1}{\sigma^2} \underbrace{Var(X)}_{\sigma^2} = 1$$

Closure of normal distribution – Under Shifting and Scaling

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Mean and variance follow from properties you know! The fact that result of shifting and scaling still normal is not obvious, but not too difficult



Normal Distribution

Paranormal Distribution