Continuous Random Variables

CSE 312 Spring 26

Lecture 15

Agenda

- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Expectation and Variance of continuous r.v.
- Introduction to continuous zoo

Often we want to model experiments where the outcome is <u>not</u> discrete.

Definition. A **continuous random variable** X has a continuous range of values that it can take (an interval or a set of intervals).

Thus, a continuous random variable can take on an uncountable set of possible values

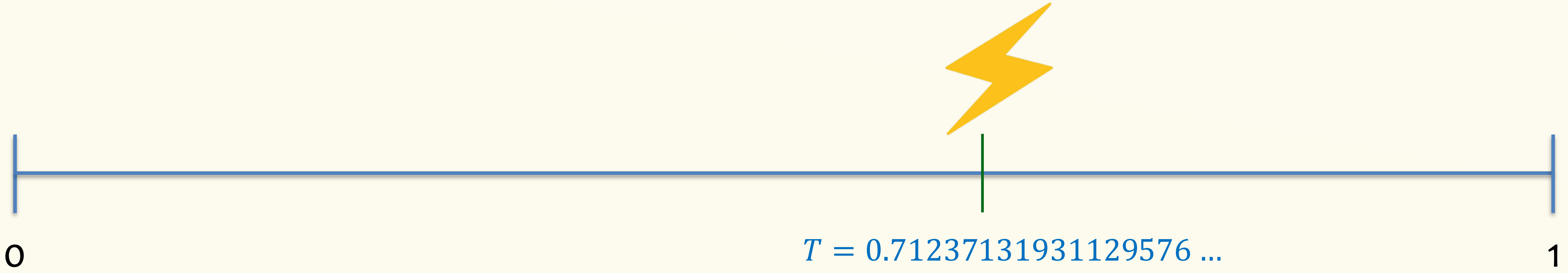
Examples:

- Time of an event
- Response time of a job
- Speed of a device
- Location of a satellite
- Distance between people’s eyeballs

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
  - – Time measured with infinitesimal precision.

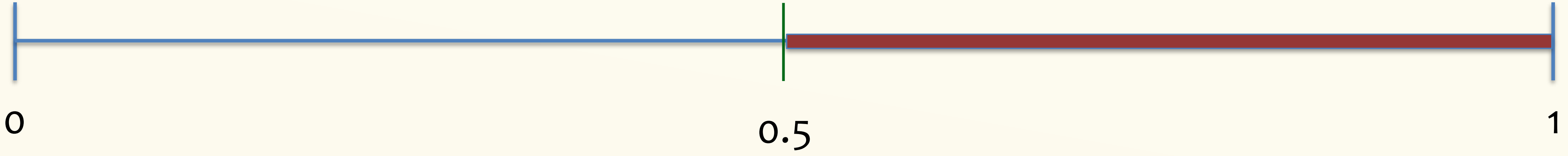


The outcome space is not discrete

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$P(T \geq 0.5) =$



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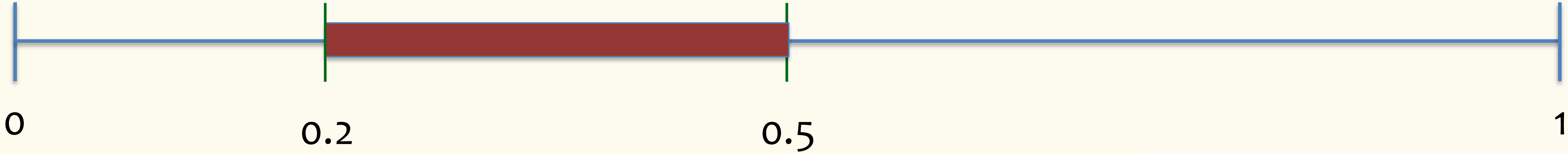

$$P(0.2 \leq T \leq 0.5) =$$

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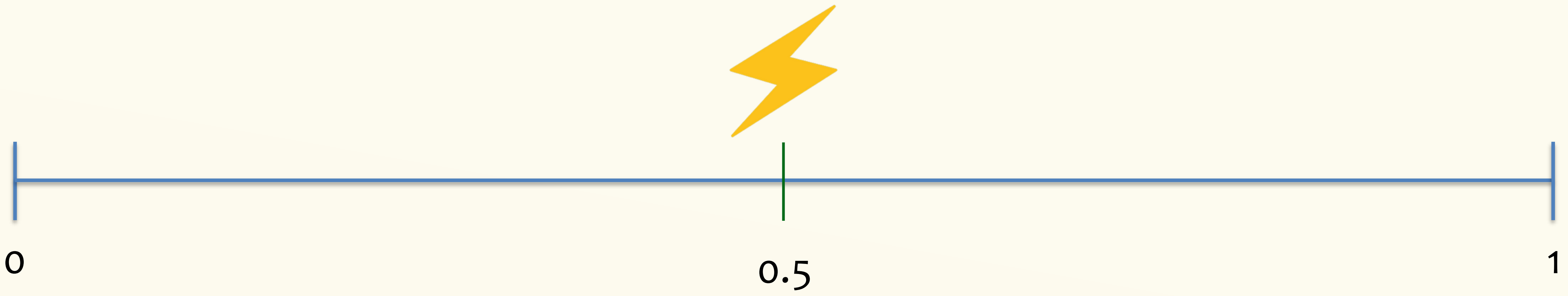

$P(T = 0.5) =$

Bottom line

- This gives rise to a different type of random variable
- $P(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
  - $P(T \in [0,1]) = 1$
  - $P(T \in [a,b]) = b - a$
  - …
- How do we model the behavior of T?

First try: A discrete approximation

Example – Lightning Strike

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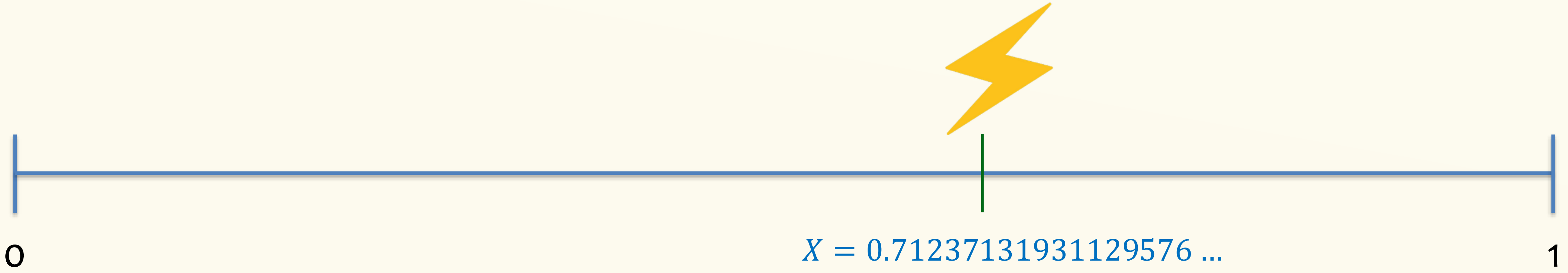
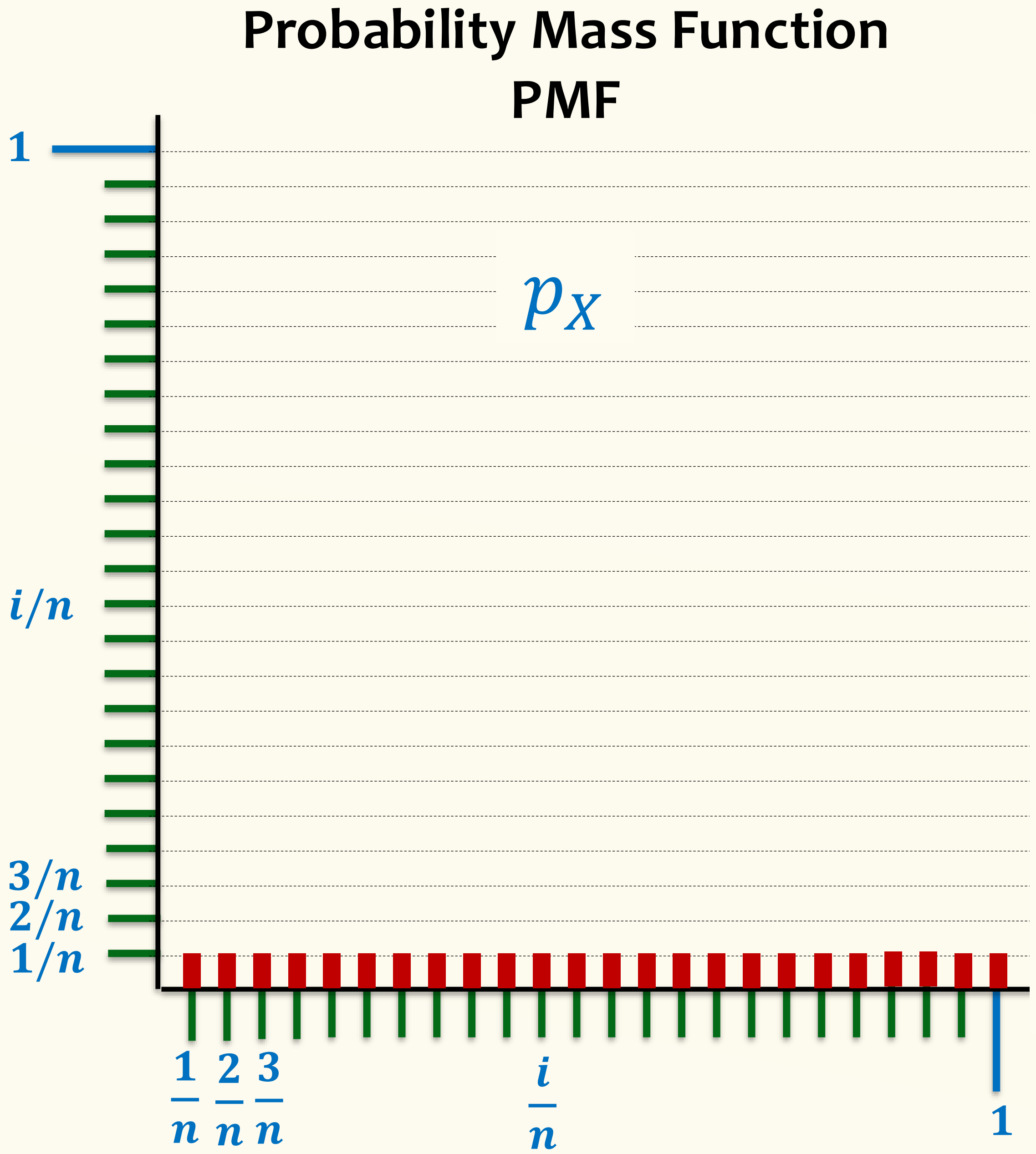
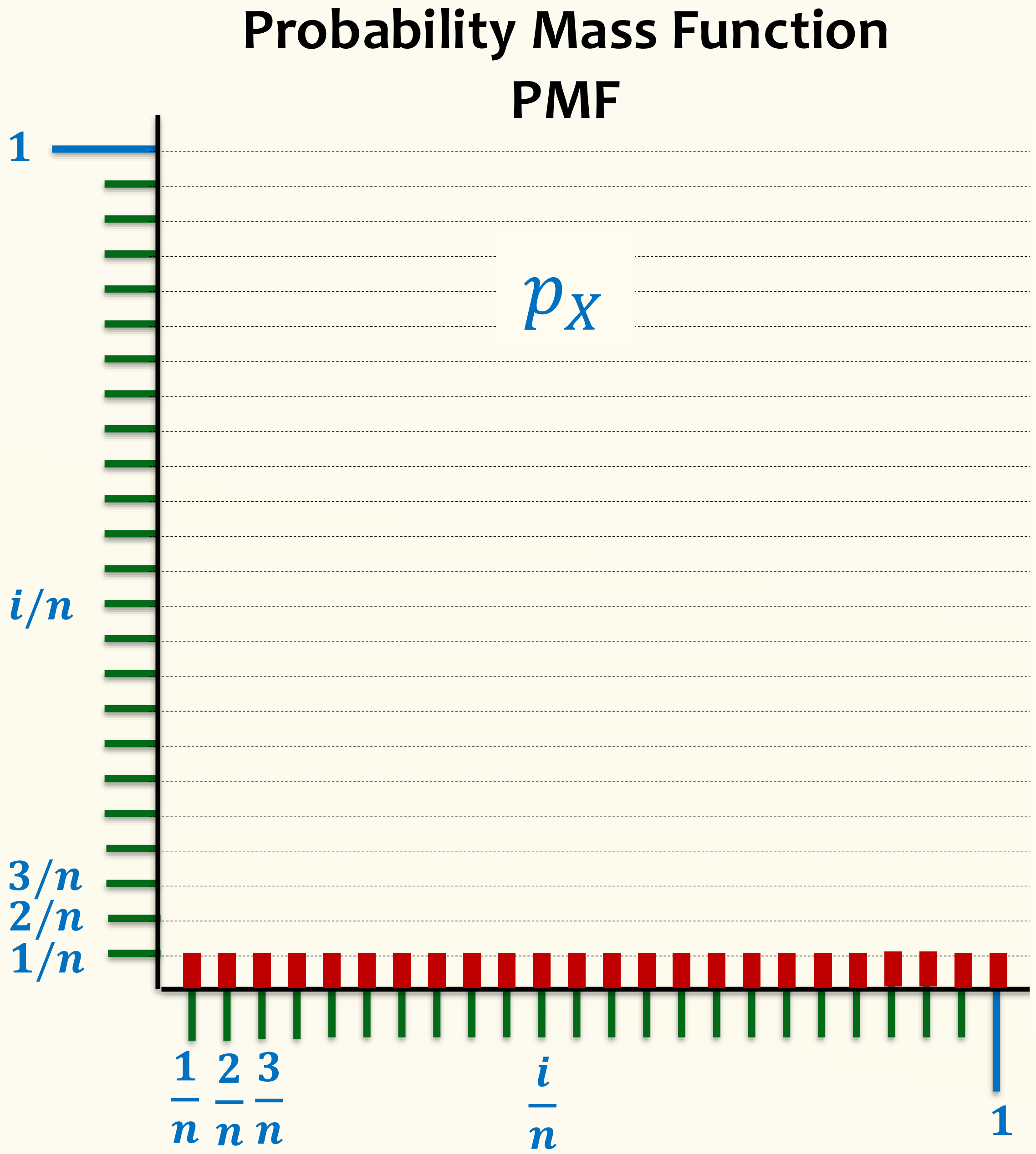


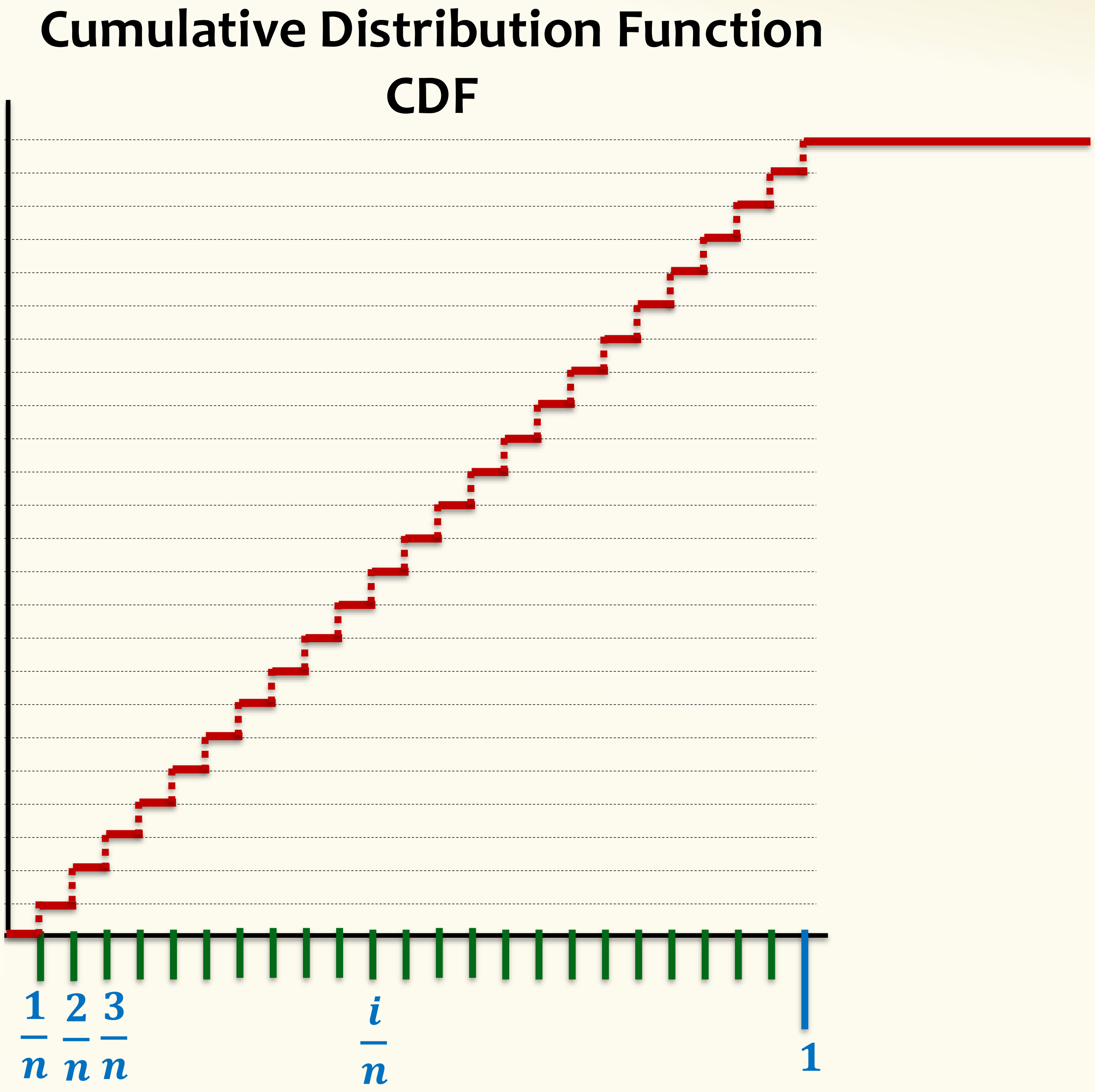
Discrete approximation?

A Discrete Approximation



A Discrete Approximation



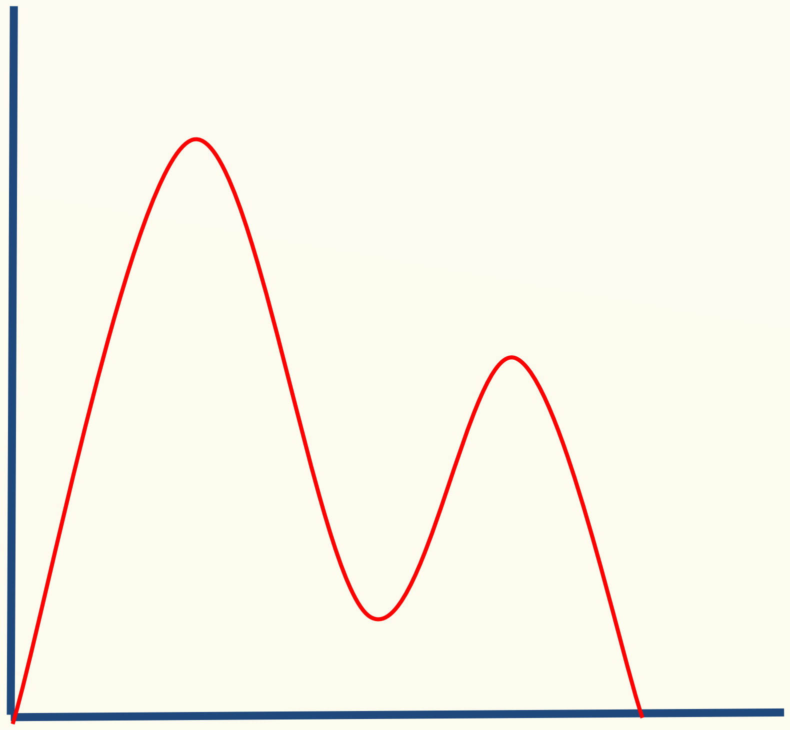


Given the CDF F_X of a random variable X with $\Omega_X = \{0, 1, 2, \ldots\}$, how do you compute the pmf?

$\Pr(X = k) =$

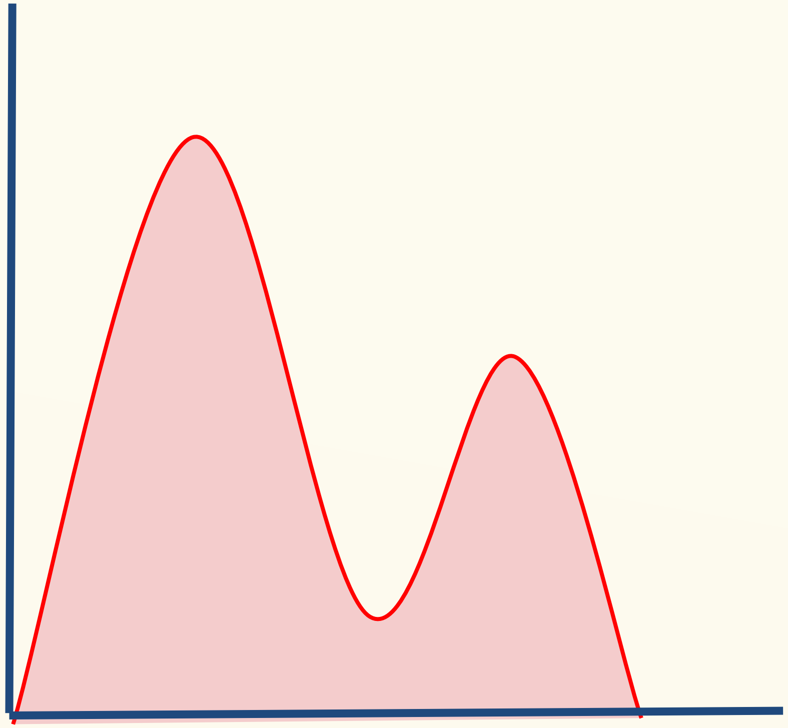
a. $F_X(k-1)$
b. $F_X(1) + F_X(2) + \cdots + F_X(k-1)$
c. $F_X(k) - F_X(k-1)$
d. I don't know.

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \to \mathbb{R}$, such that



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Probability Density Function - Intuition



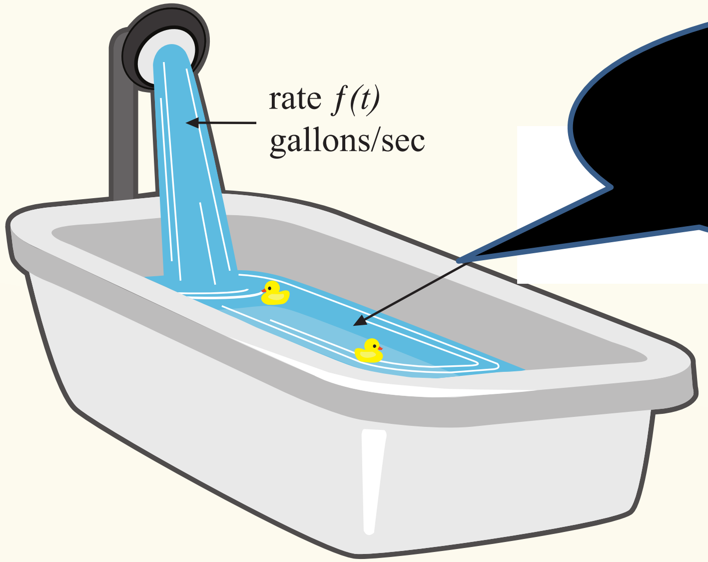
Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

Density as a rate

Density functions are not necessarily related to probability.

Example: Filling a bathtub at rate $f_X(t) = t^2$ gallons/sec, where $t \geq 0$

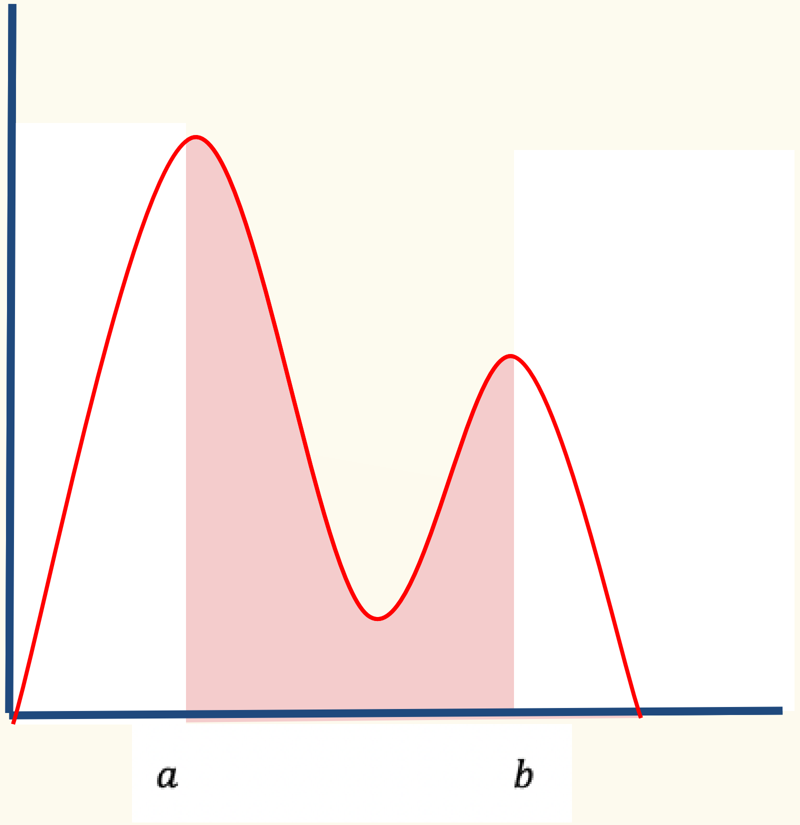


Q: How much water after 4 seconds?

A: $$\int_0^4 f_X(t)dt = \int_0^4 t^2 dt = \frac{64}{3} \quad \text{gallons}$$

Q: Is $f_X(t)$ a p.d.f.?

Probability Density Function - Intuition



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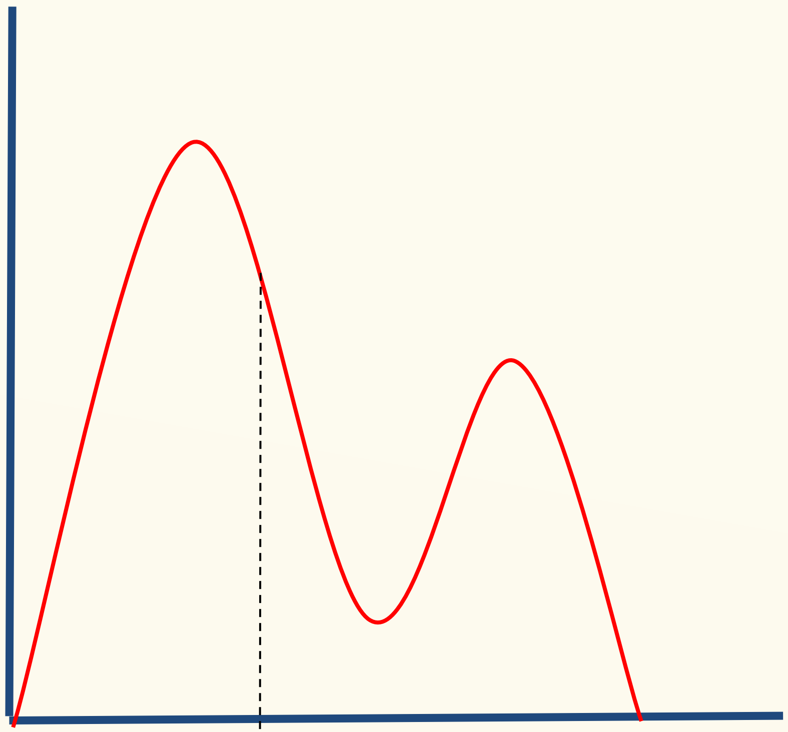

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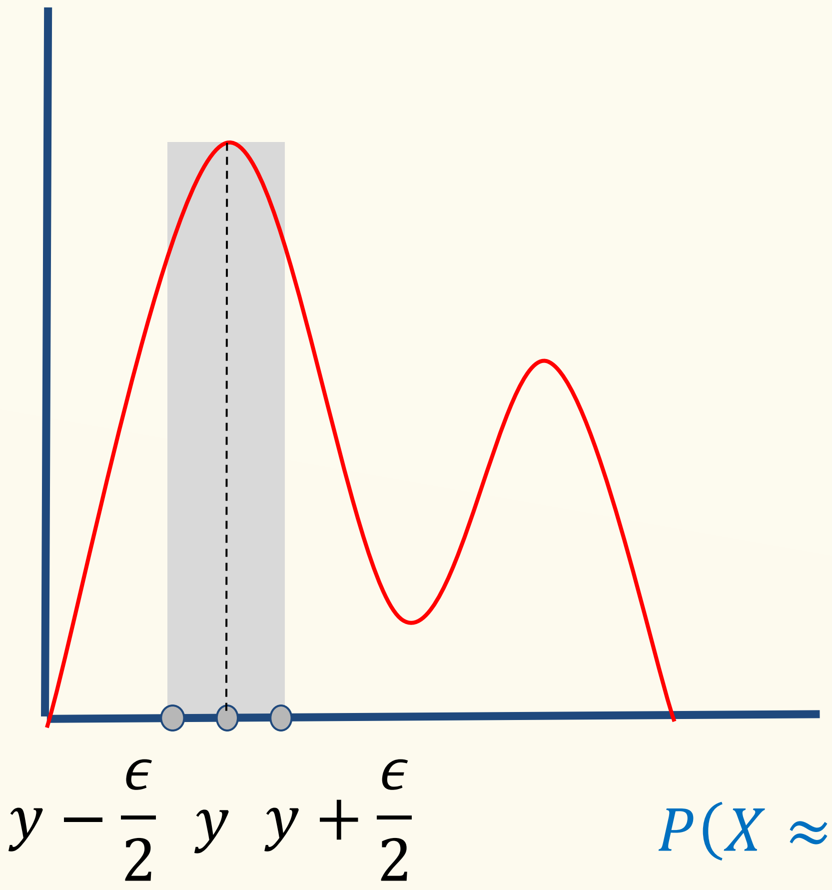
$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

Density ≠ Probability

$f_X(y) \neq 0 \qquad P(X = y) = 0$

Probability Density Function - Intuition



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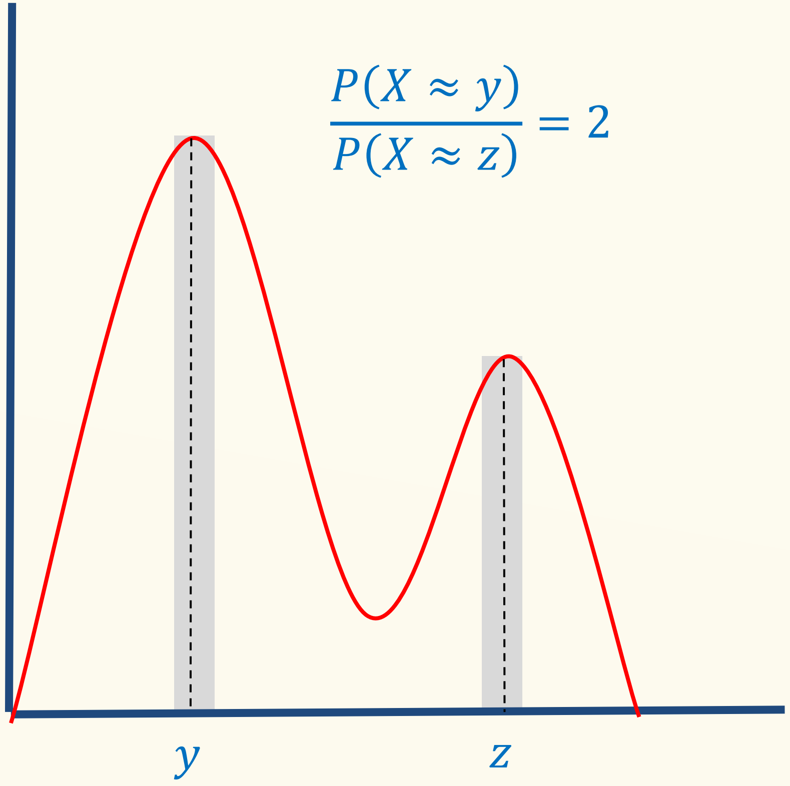
$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

What $f_X(x)$ measures: The local ***rate*** at which probability accumulates

Probability Density Function - Intuition



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Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F_X(x)$

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By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F_X(x)$

Therefore: $P(X \in [a, b]) = F_X(b) - F_X(a)$

F_X is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

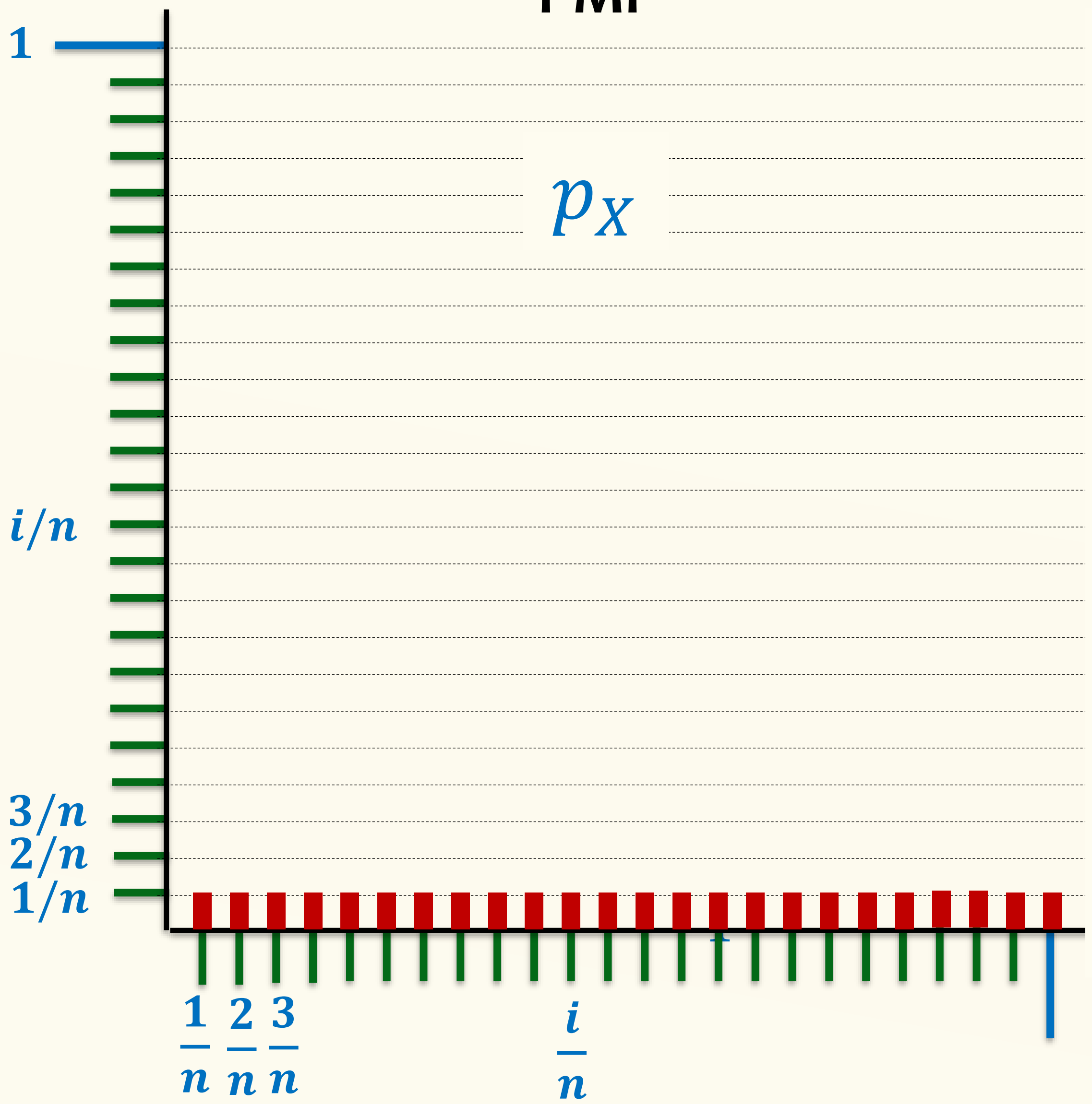
$\lim_{a \to -\infty} F_X(a) = P(X \leq -\infty) = 0 \qquad \lim_{a \to +\infty} F_X(a) = P(X \leq +\infty) = 1$

From Discrete to Continuous

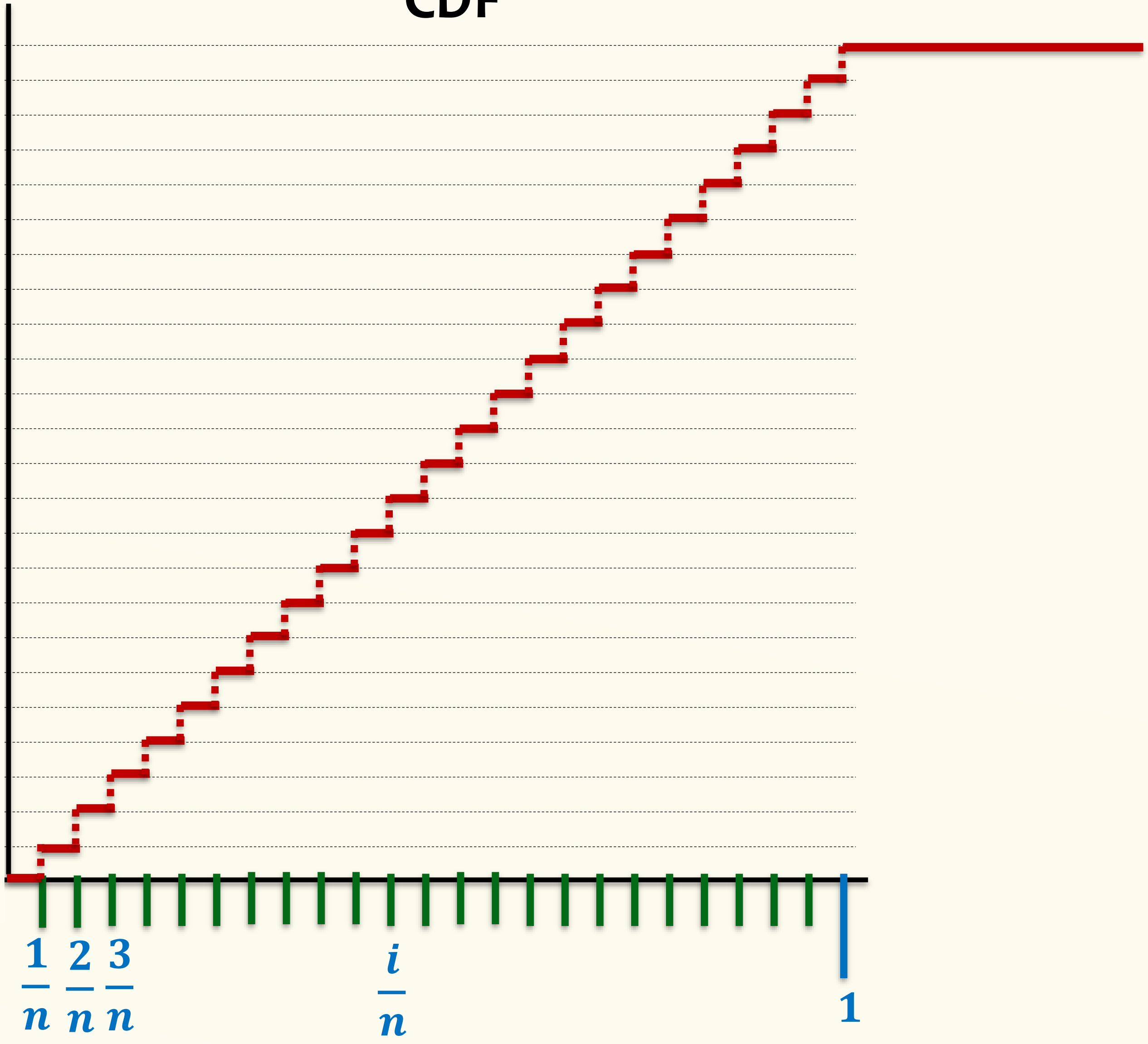
	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_{x} p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

A Discrete Approximation

Probability Mass Function
PMF



Cumulative Distribution Function
CDF

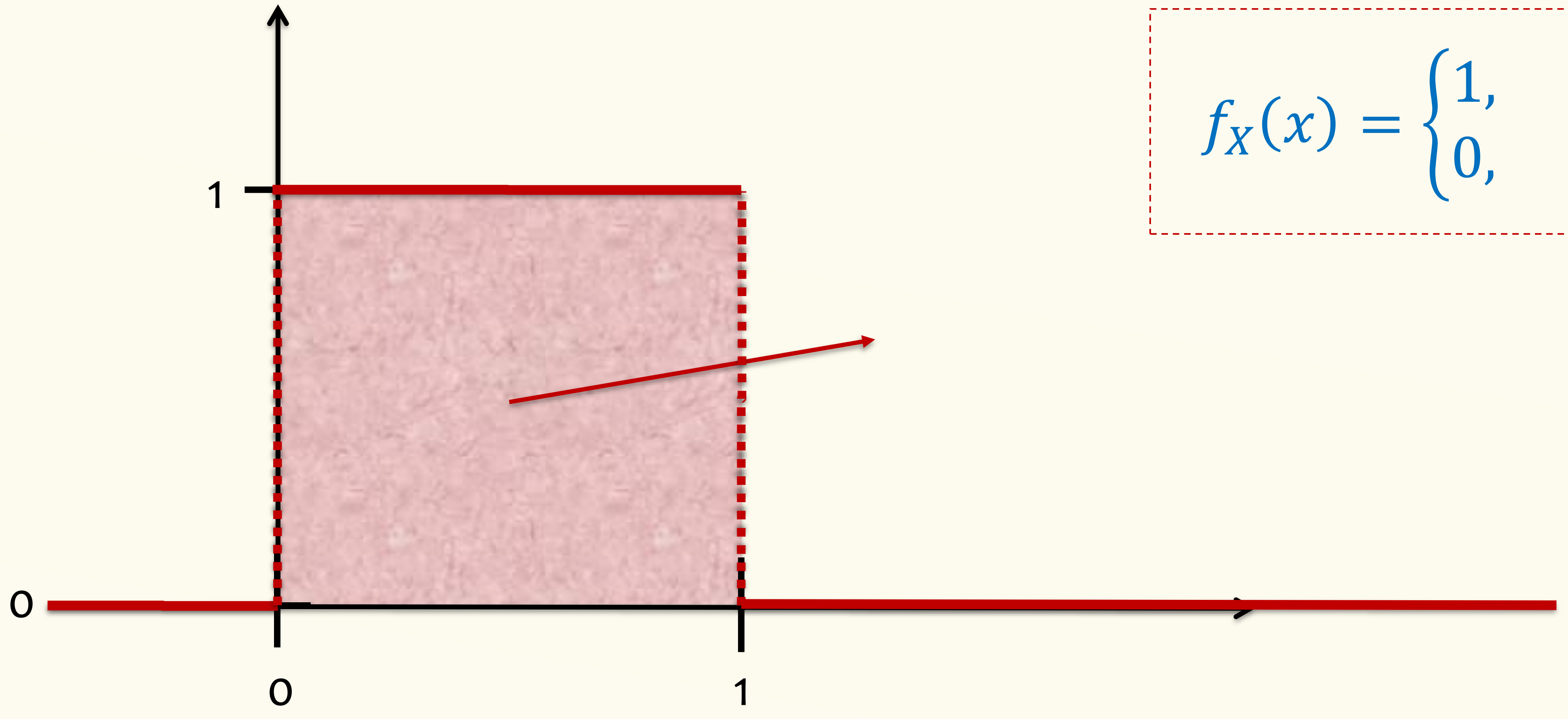


PDF of Uniform RV

$X \sim \text{Unif}(0,1)$

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



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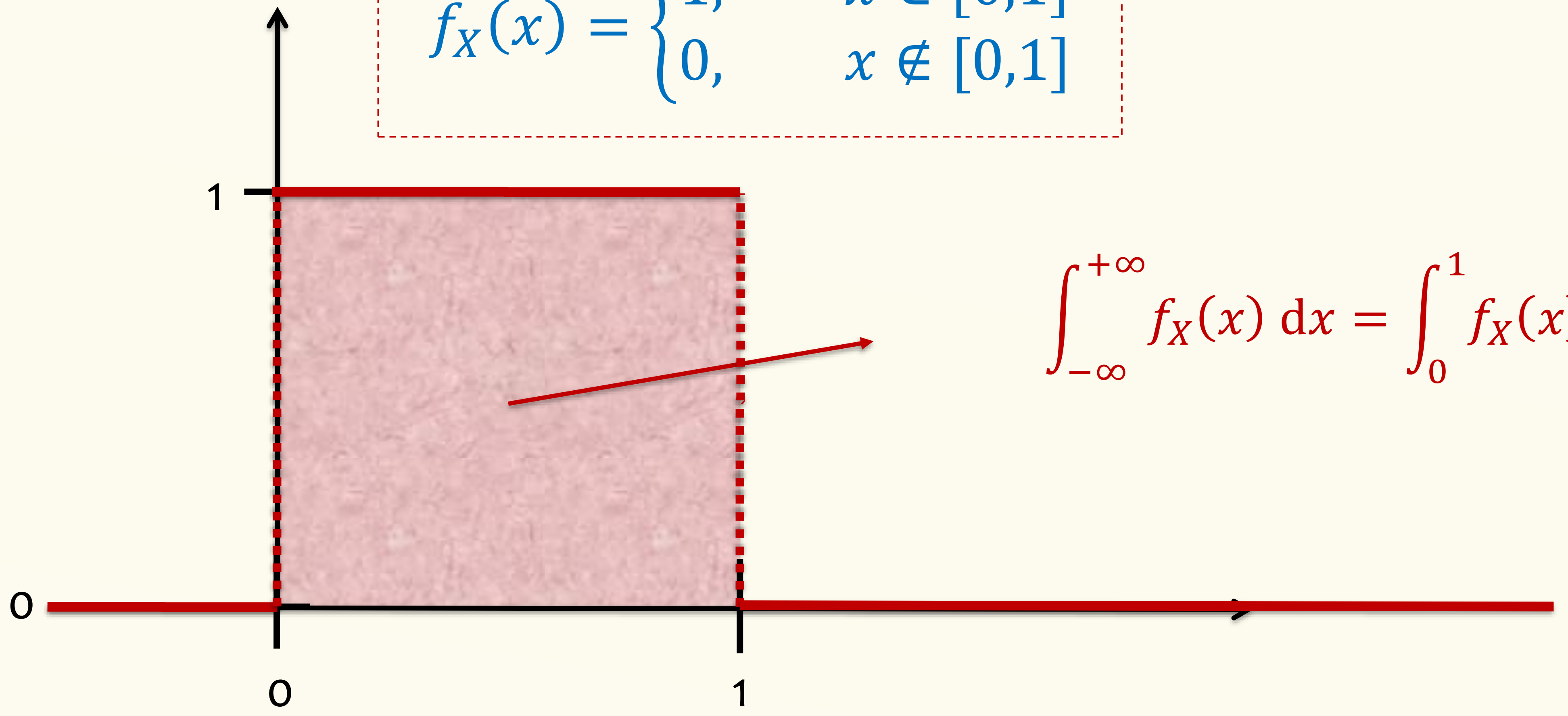

$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 1 \cdot 1 = 1$$

Probability of Event

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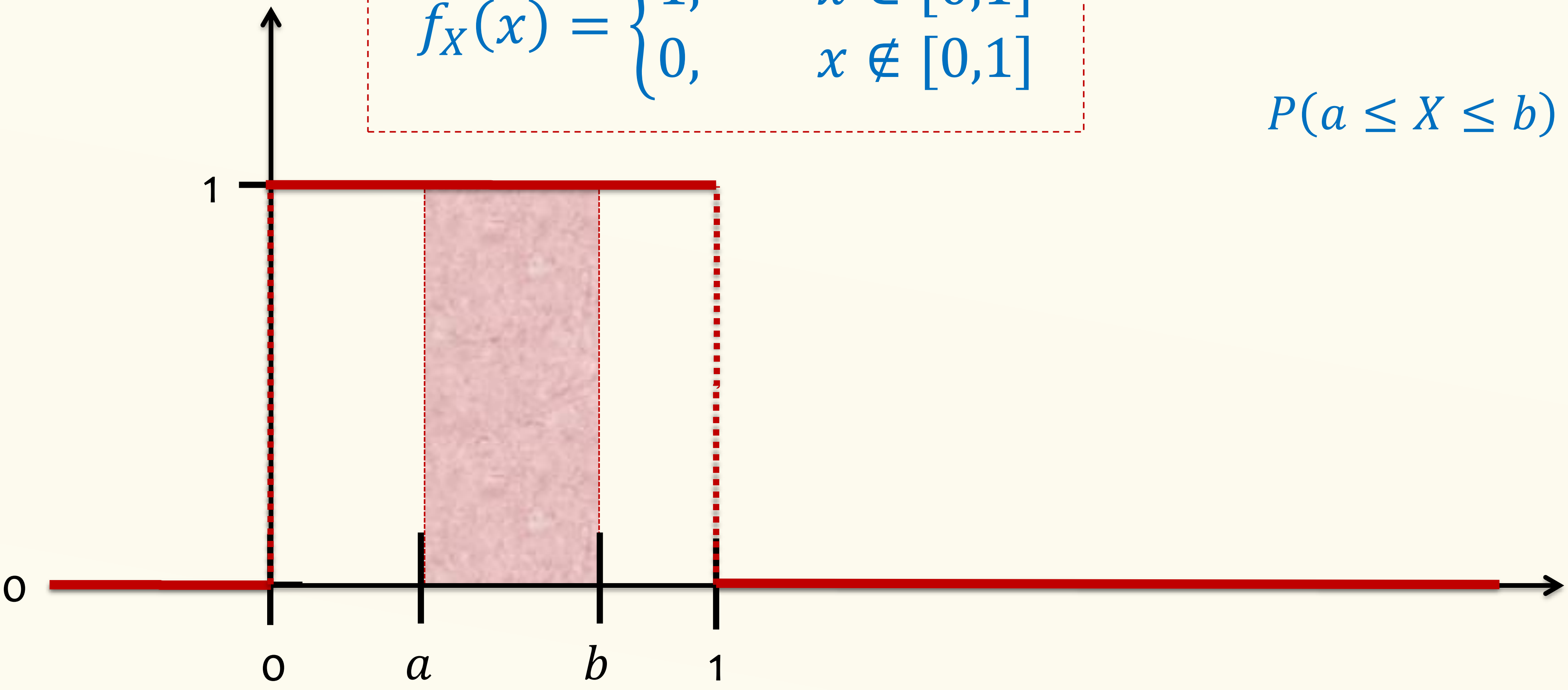
$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$



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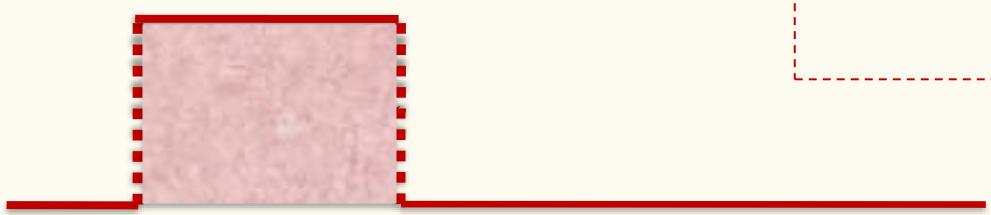

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PDF of Uniform RV

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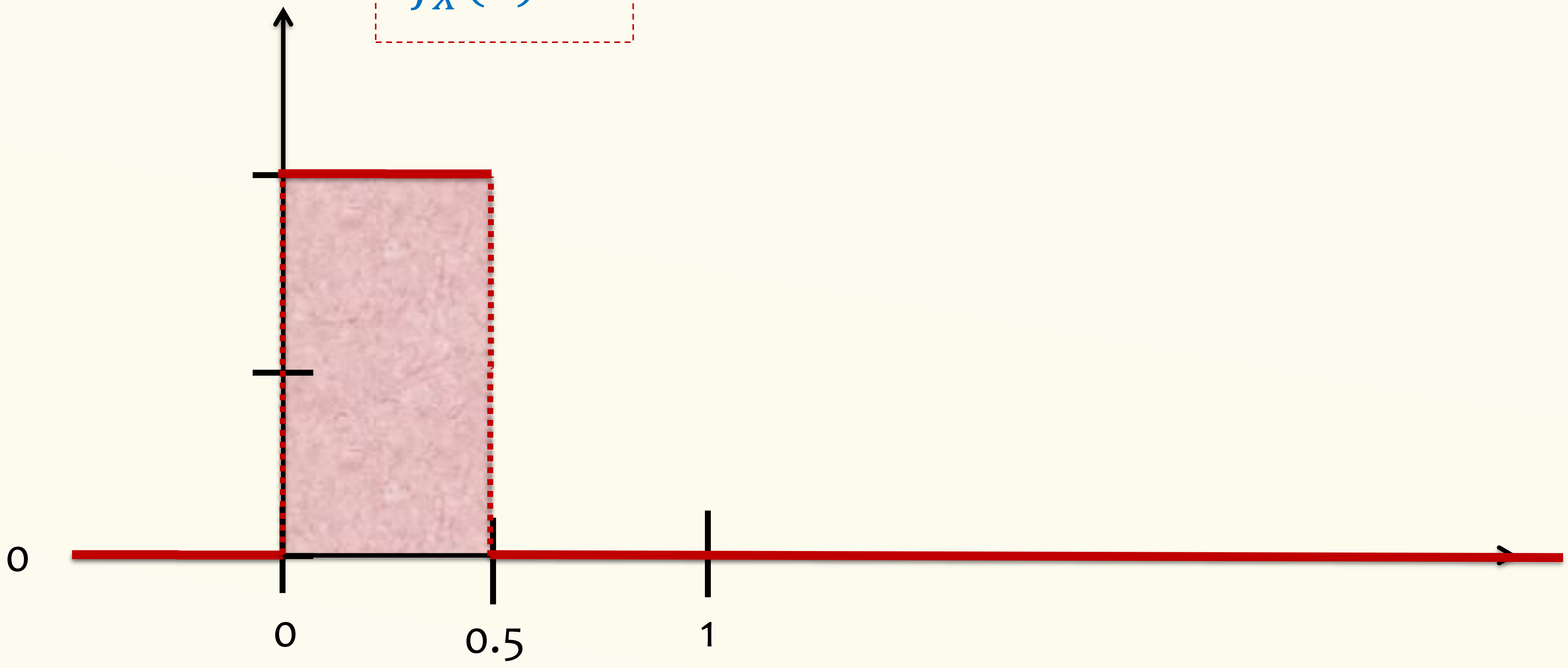


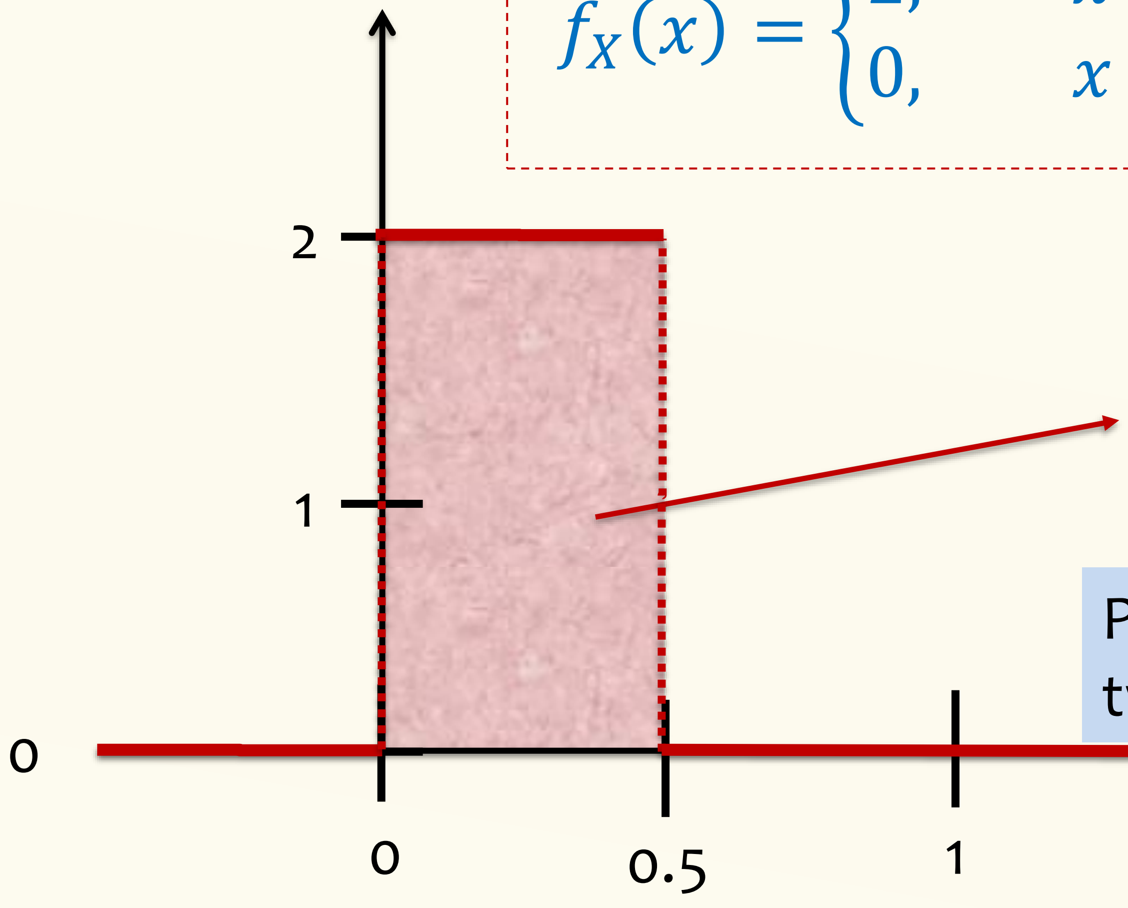
PDF of Uniform RV

Density $\neq$ Probability

$f_X(x) \gg 1$ is possible!

$X \sim \mathrm{Unif}(0,0.5)$

$$f_X(x) = \begin{cases} 2, & x \in [0,0.5] \\ 0, & x \notin [0,0.5] \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 2 \cdot 0.5 = 1$$

Probability on $[0,0.5]$ accumulates at twice the rate compared to $\mathrm{Unif}(0,1)$

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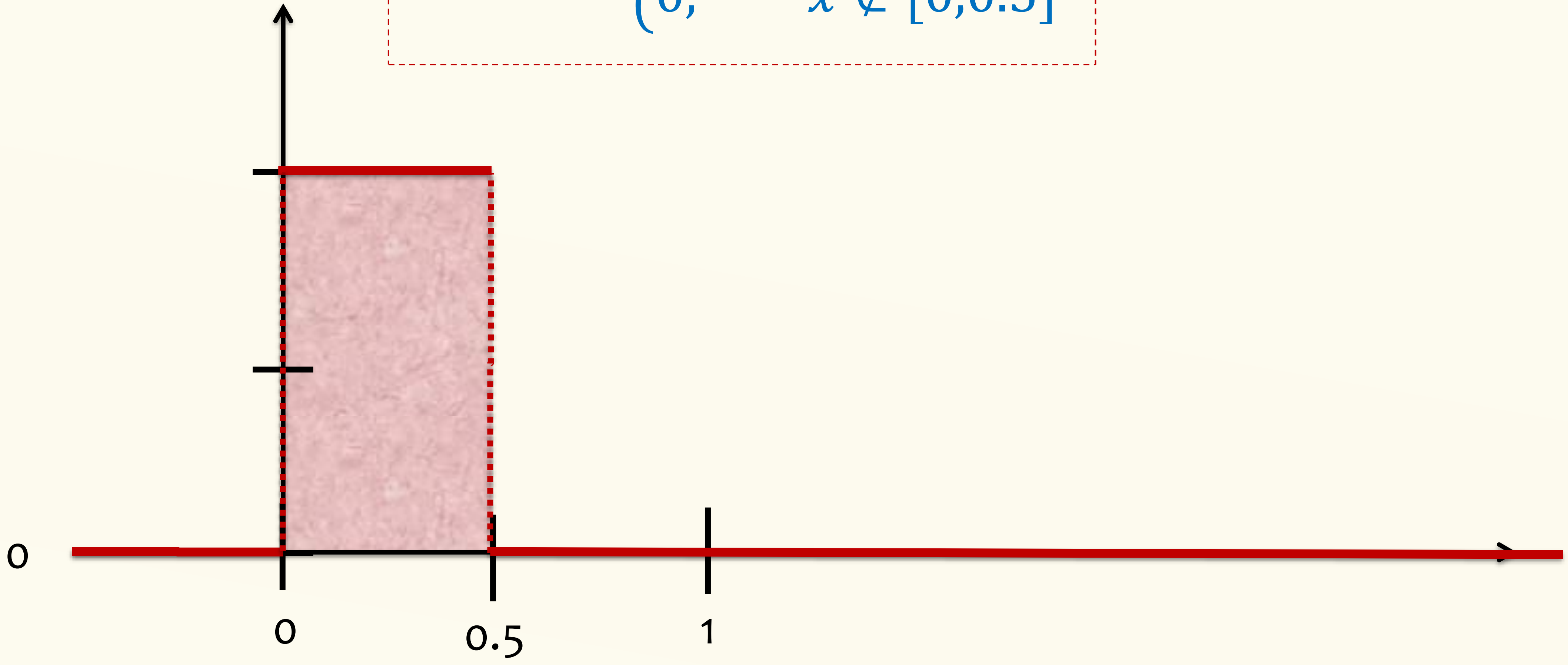
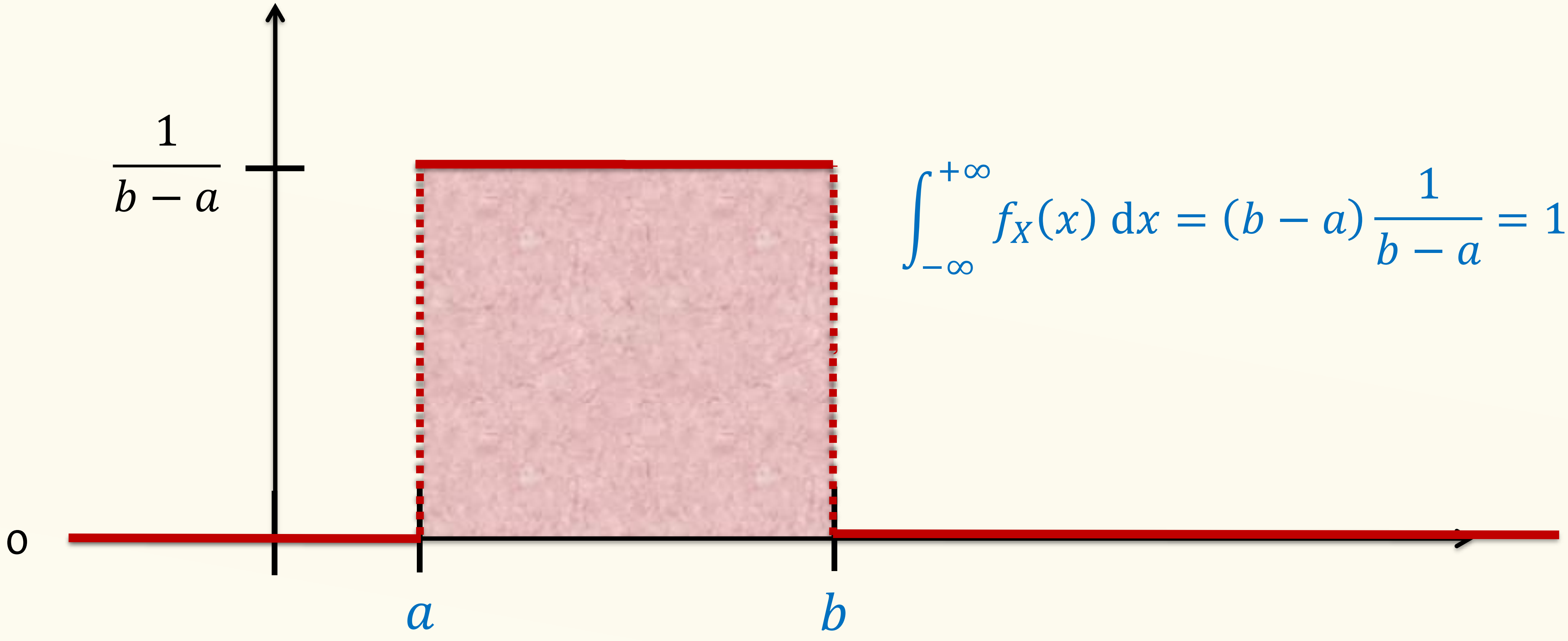


Uniform Distribution

$X \sim \text{Unif}(a, b)$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = (b-a)\frac{1}{b-a} = 1$$

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Expectation of a Continuous RV

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Proof follows same ideas as discrete case

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Proofs follow same ideas as discrete case

Definition. The **variance** of a continuous RV X is defined as

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}[X])^2 \, \mathrm{d}x = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Expectation, Variance, and LOTUS

Defn: For a continuous r.v. X with p.d.f. $f_X(\cdot)$, we have:

$$\boldsymbol{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$$

$$\boldsymbol{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)dx$$

$$\boldsymbol{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx$$

$$\boldsymbol{Var}(X) = \boldsymbol{E}[(X - \boldsymbol{E}[X])^2\ ] = \int_{-\infty}^{\infty} (x\ - \boldsymbol{E}[X])^2 f_X(x)dx = \boldsymbol{E}[X^2] - \boldsymbol{E}[X]^2$$

Agenda

- Zoo of continuous random variables
  - Uniform Distribution
  - Exponential Distribution
  - Normal Distribution

Expectation of a Continuous RV

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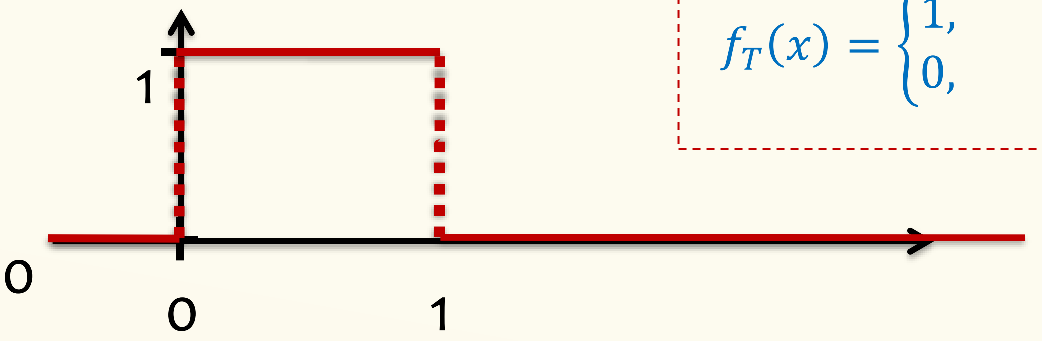
Example. $T \sim \mathrm{Unif}(0,1)$



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Expectation of a Continuous RV

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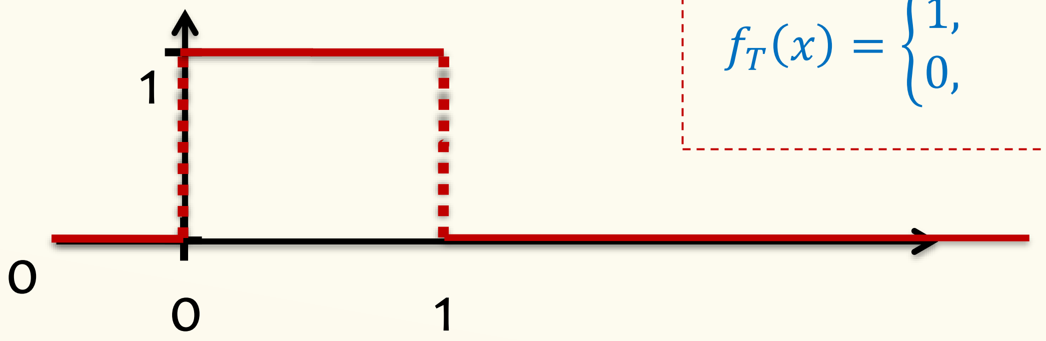

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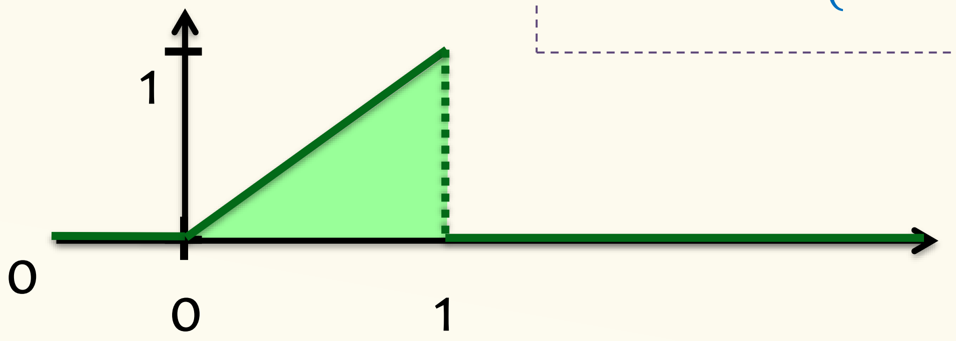

$$f_T(x) \cdot x = \begin{cases} x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

$$\mathbb{E}[X] = \underbrace{\frac{1}{2} 1^2}_{\text{Area of triangle}} = \frac{1}{2}$$

Uniform Density – Expectation

$$X \sim \mathrm{Unif}(a, b)$$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

$$= \frac{1}{b-a} \int_a^b x \, \mathrm{d}x \quad = \frac{1}{b-a} \left(\frac{x^2}{2} \right) \Bigg|_a^b = \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right)$$

$$= \frac{(b-a)(a+b)}{2(b-a)} \quad = \frac{a+b}{2}$$

Uniform Density – Variance

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

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$\mathbb{E}[X^2] =$

Uniform Density – Variance

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$X \sim \mathrm{Unif}(a,b)$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \,\mathrm{d}x$$

$$= \frac{1}{b-a}\int_a^b x^2 \,\mathrm{d}x = \frac{1}{b-a}\left(\frac{x^3}{3}\right)\Bigg|_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

Uniform Density – Variance

$$\mathbb{E}[X^2] = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}[X] = \frac{a+b}{2}$$

$X \sim \text{Unif}(a, b)$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

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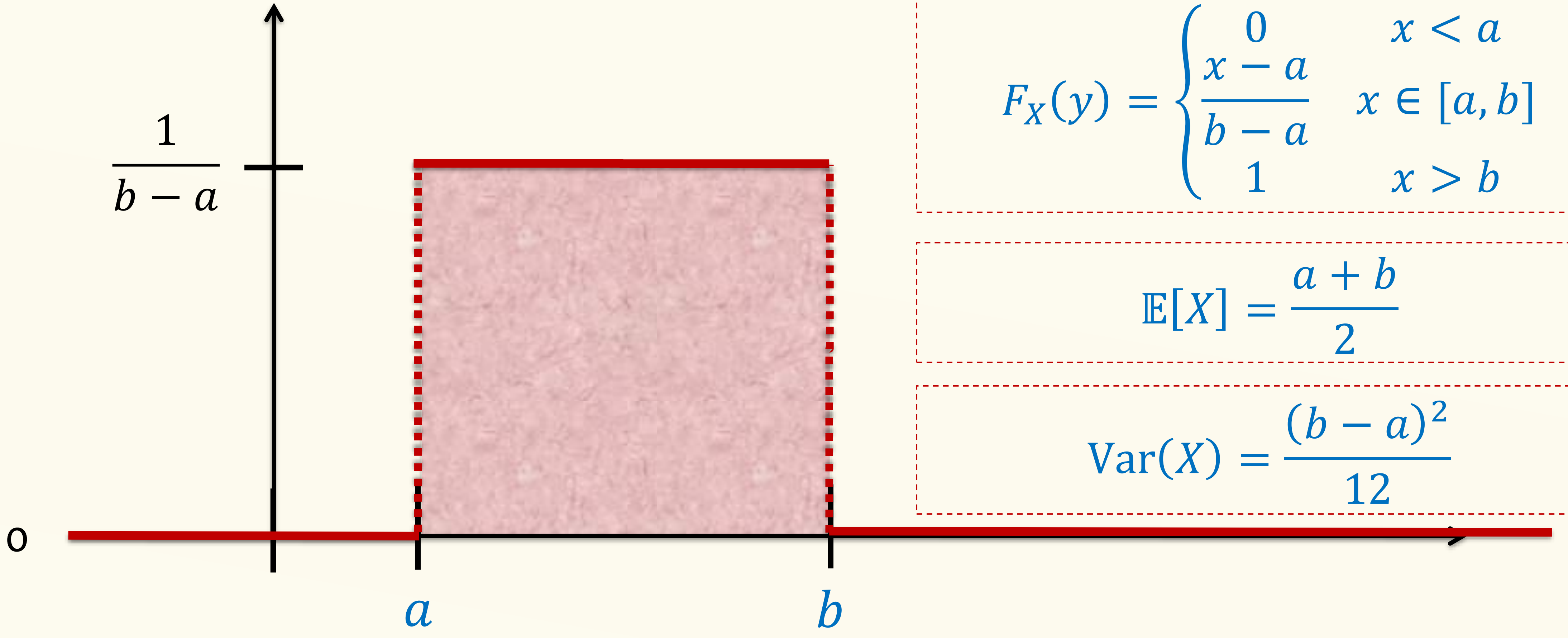
$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}$$

Uniform Distribution Summary

$X \sim \text{Unif}(a, b)$



$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

Agenda

- Zoo of continuous random variables
  - Uniform Distribution
  - Exponential Distribution

Recall Poisson

Assume expected # of occurrences of an event per unit of time is λ (independently)

- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER
- Rate of radioactive decay

Numbers of occurrences of event in one unit of time: Poisson distribution

$$P(W = i) = e^{-\lambda}\frac{\lambda^i}{i!}$$ (Discrete)

How long to wait until next event? Exponential density!

Let's define it and then derive it!

Exponential Density - Warmup

$W \sim Poi(\lambda) \Rightarrow P(W = i) = e^{-\lambda} \frac{\lambda^i}{i!}$

Assume expected # of occurrences of an event per unit of time is λ (independently)

What is $\mathbb{E}[Z_t]$ where Z_t = # occurrences of event per t units of time?

Exponential Density - Warmup

$W \sim Poi(\lambda) \Rightarrow P(W = i) = e^{-\lambda} \frac{\lambda^i}{i!}$

Assume expected # of occurrences of an event per unit of time is λ (independently)

What is the distribution of Z_t = # occurrences of event per t units of time?

$$\mathbb{E}[Z_t] = t\lambda$$

Z_t is independent over disjoint intervals

So $Z_t \sim Poi(t\lambda)$

$W \sim Poi(\lambda) \Rightarrow P(W = i) = e^{-\lambda} \frac{\lambda^i}{i!}$

The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is λ (independently)

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- Let X be the time till the first event. We will compute $F_X(t)$ and $f_X(t)$
- We know $Z_t \sim Poi(t\lambda)$ is the # of events in the first t units of time, for $t \geq 0$.

$W \sim Poi(\lambda) \Rightarrow P(W = i) = e^{-\lambda} \frac{\lambda^i}{i!}$

The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is λ (independently)

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0,1,2, \ldots\}$
- Let $X \sim Exp(\lambda)$ be the time till the first event. We will compute $F_X(t)$ and $f_X(t)$
- We know $Z_t \sim Poi(t\lambda)$ be the # of events in the first t units of time, for $t \geq 0$.
- $P(X > t) = P(\text{no event in the first } t \text{ units}) = P(Z_t = 0) = e^{-t\lambda} \frac{(t\lambda)^0}{0!} = e^{-t\lambda}$
- $F_X(t) = P(X \leq t) = 1 - P(Y > t) = 1 - e^{-t\lambda}$
- $f_X(t) = \frac{d}{dt} F_X(t) = \lambda e^{-t\lambda}$

$P(X > t) = e^{-t\lambda}$

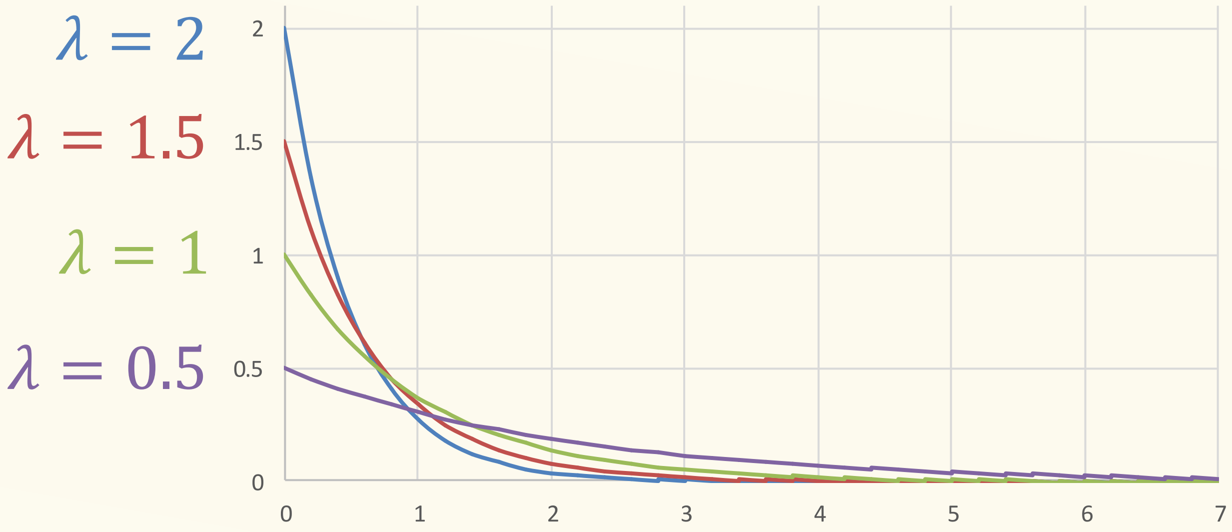
Exponential Distribution

Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$



Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P(X > t) = e^{-t\lambda}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P(X > t) = e^{-t\lambda}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \,\mathrm{d}x$$

$$= \int_{0}^{+\infty} \lambda e^{-\lambda x} \cdot x \,\mathrm{d}x$$

$$= \left(-(x + \frac{1}{\lambda}) e^{-\lambda x} \right) \Big|_0^{\infty} = \frac{1}{\lambda}$$

Somewhat complex calculation
use integral by parts

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

$P(X > t) = e^{-t\lambda}$

Exponential Distribution

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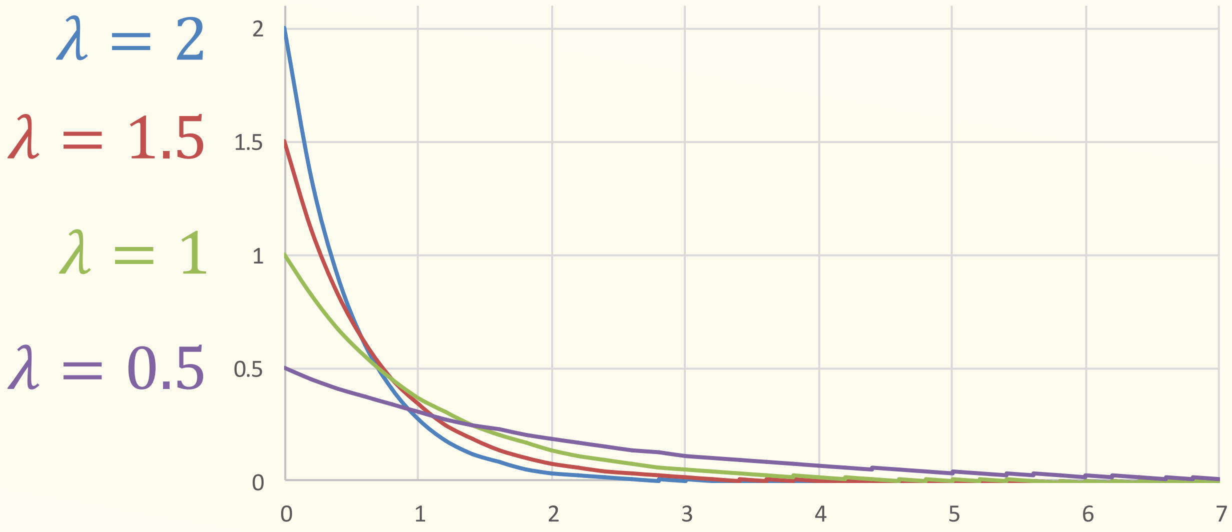


Memorylessness

Definition. A random variable is **memoryless** if for all $s, t > 0$,

$$P(X > s + t \mid X > s) = P(X > t).$$

Fact. $X \sim \mathrm{Exp}(\lambda)$ is memoryless.

Assuming an exponential distribution, if you've waited s minutes,
The probability of waiting t more is exactly same as when $s = 0$.

Memorylessness of Exponential

$P(X > t) = e^{-\lambda t}$

Proof that assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as when $s = 0$

Fact. $X \sim \mathrm{Exp}(\lambda)$ is memoryless.

Proof.

$P(X > s + t \mid X > s) =$

Memorylessness of Exponential

$P(X > t) = e^{-\lambda t}$

Proof that assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as when $s = 0$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$$P(X > s + t \mid X > s) = \frac{P(\{X > s + t\} \cap \{X > s\})}{P(X > s)}$$

$$= \frac{P(X > s + t)}{P(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous)

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If there is one person currently being served, what is the probability that you will have to wait between 10 and 20 mins until you start getting service?

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If there is one person currently being served, what is the probability that you will have to wait between 10 and 20 mins until you start getting service?

$$T \sim Exp(\frac{1}{10})$$

$$P(10 \le T \le 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}}\, dx$$

$y = \frac{x}{10}$ so $dy = \frac{dx}{10}$

$$P(10 \le T \le 20) = \int_{1}^{2} e^{-y}\, dy = -e^{-y}\Big|_1^2 = e^{-1} - e^{-2}$$

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If there is one person currently being served, what is the probability that you will have to wait between 10 and 20 mins until you start getting service?

$$T \sim Exp(\frac{1}{10})$$

so $F_T(t) = 1 - e^{-\frac{t}{10}}$

$$P(10 \leq T \leq 20) = F_T(20) - F_T(10)$$
$$= 1 - e^{-\frac{20}{10}} - \left(1 - e^{-\frac{10}{10}}\right) = e^{-1} - e^{-2}$$

Recap – Continuous RVs

Probability Density Function (PDF).

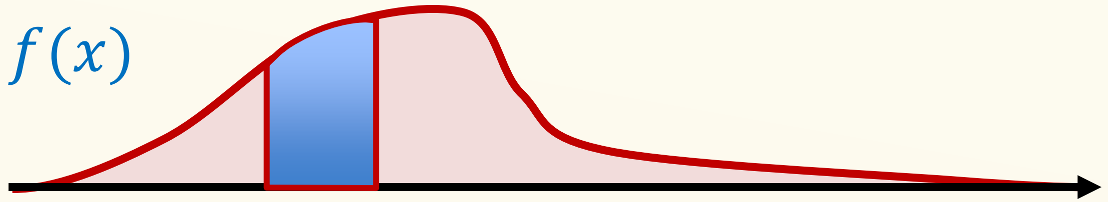
$f: \mathbb{R} \to \mathbb{R}$ s.t.

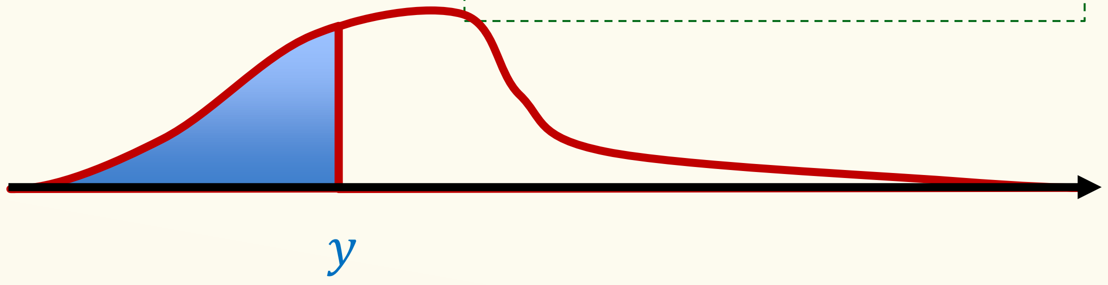
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = 1$

Cumulative Distribution Function (CDF).

$$F(y) = \int_{-\infty}^{y} f(x)\, \mathrm{d}x$$

Theorem. $f(x) = \frac{dF(x)}{dx}$





Density ≠ Probability !

$$P(X \in [a,b]) = \int_a^b f_X(x)\mathrm{d}x$$
$$= F_X(b) - F_X(a)$$

$$F_X(y) = P(X \leq y)$$

Recap: From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_{x} p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_{x} g(x)\, p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$