

The Poisson Distribution

CSE 312 Spring 26
Lecture 13

Discrete Uniform Random Variables (2)

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is **uniform**.

Notation: $X \sim \text{Unif}(a, b)$

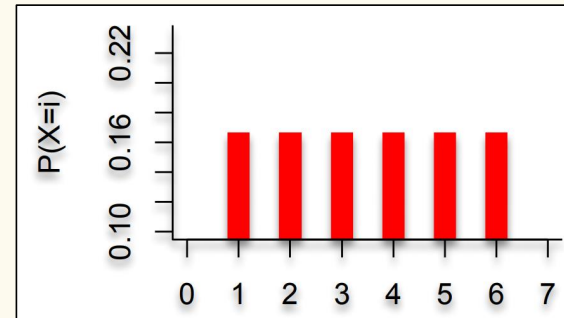
PMF: $P(X = i) = \frac{1}{b - a + 1}$

Expectation: $\mathbb{E}[X] = \frac{a+b}{2}$

Variance: $\text{Var}(X) = \frac{(b-a)(b-a+1)}{12}$

Example: value shown on one roll of a fair die is $\text{Unif}(1,6)$:

- $P(X = i) = 1/6$
- $\mathbb{E}[X] = 7/2$
- $\text{Var}(X) = 35/12$



Bernoulli Random Variables (2)

Indicator

A random variable X that takes value 1 (“Success”) with probability p , and 0 (“Failure”) otherwise. X is called a **Bernoulli random variable**.

Notation: $X \sim \text{Ber}(p)$

PMF: $P(X = 1) = p, P(X = 0) = 1 - p$

Expectation: $\mathbb{E}[X] = p$ Note: $\mathbb{E}[X^2] = p$

Variance: $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = p(1 - p)$

Examples:

- Coin flip
- Randomly guessing on a MC test question
- A server in a cluster fails
- Whether or not a share of a particular stock pays off or not
- Any indicator r.v.

Binomial Random Variables (3)

A discrete random variable $X = \sum_{i=1}^n Y_i$ where each $Y_i \sim \text{Ber}(p)$.

Counts number of successes in n independent trials, each with probability p of success.

X is a **Binomial random variable**

Notation: $X \sim \text{Bin}(n, p)$

PMF: $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$

Expectation: $\mathbb{E}[X] = np$

Variance: $\text{Var}(X) = np(1 - p)$

Examples:

- # of heads in n indep coin flips
- # of 1s in a randomly generated n bit string
- # of servers that fail in a cluster of n computers
- # of bit errors in file written to disk
- # of elements in a bucket of a large hash table
- # of n different stocks that “pay off”

$\{0, 1, \dots, n\}$

Geometric Random Variables (2)

A discrete random variable X that models the number of independent trials $Y_i \sim \text{Ber}(p)$ up until and including the first success.

X is called a **Geometric random variable** with parameter p .

Notation: $X \sim \text{Geo}(p)$

PMF: $P(X = k) = (1 - p)^{k-1} p$
 $k=1, 2, \dots$

Expectation: $\mathbb{E}[X] = \frac{1}{p}$

Variance: $\text{Var}(X) = \frac{1-p}{p^2}$

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it
- # hash trials until a miner successfully mines a Bitcoin

Today

- Poisson Random Variables

Preview: Poisson

Model: X is # events that occur in an hour

- Events occur at a fixed rate (3 per hour), but at random times.
- The expected number of events in t hours, is $3t$
- Occurrence of events on disjoint time intervals is independent

Example – Modelling car arrivals at an intersection

X = # of cars passing through a light in 1 hour

How to model?

$$X_{[0,1]} \sim \text{Bin}\left(\frac{n}{2}, p\right)$$

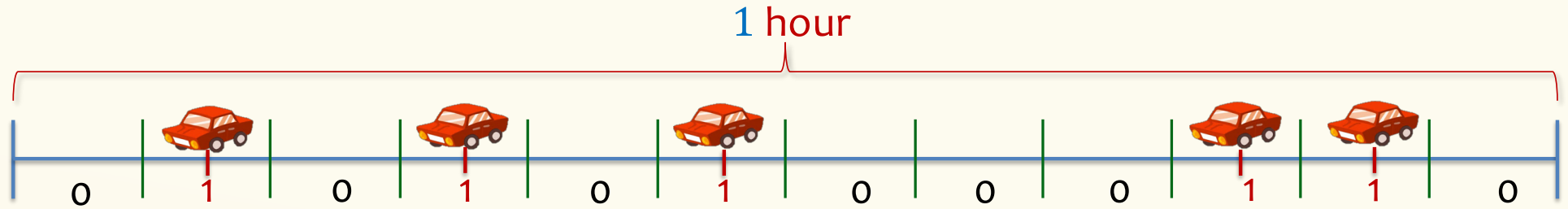
$$\frac{n}{2} \cdot \frac{p}{2} = \frac{3}{2}$$

$$\text{Bin}\left(\frac{n}{2}, p\right)$$

X = # cars passing through an intersection in 1 hour. Constant rate of arrival.

Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = 3$



$$Y_i = \begin{cases} 1 \\ 0 \end{cases}$$

y car arrives in period $\left[\frac{i-1}{n}, \frac{i}{n}\right)$

$$Y_i \sim \text{Ber}(p)$$

n intervals

p

$$X = \sum_{i=1}^n Y_i \sim \text{Bin}(n, p)$$

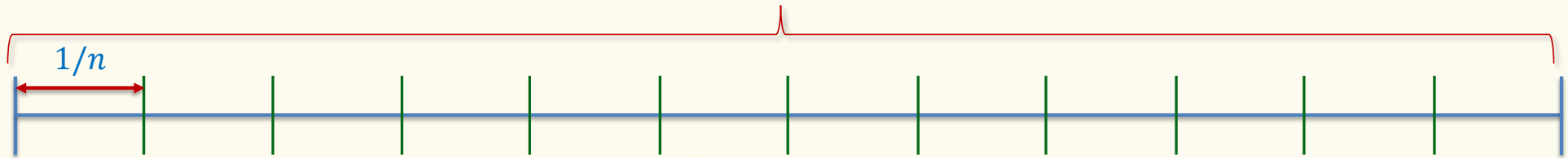
$$\mathbb{E}(X) = np = \frac{3}{2}$$

Example – Model the process of cars passing through a light in 1 hour

X = # cars passing through a light in 1 hour. $\mathbb{E}[X] = 3$

Assume: Occurrence of events on disjoint time intervals is independent

Approximation idea: Divide hour into n intervals of length $1/n$



This gives us n independent intervals

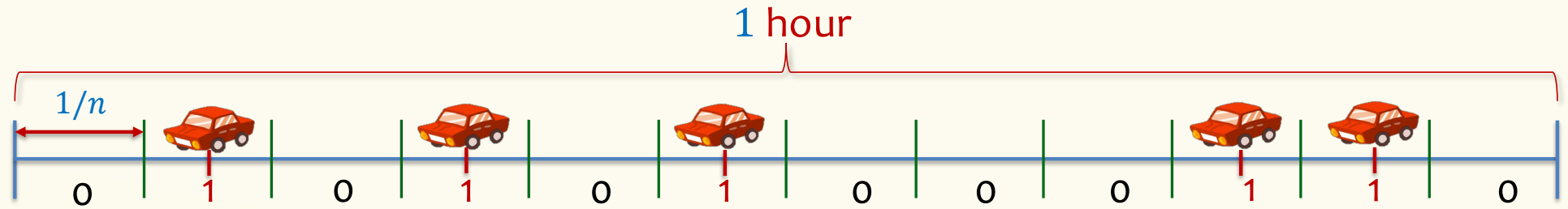
Assume either zero or one car per interval

p = probability car arrives in a single interval

Model as Binomial

X = # cars passing through a light in 1 hour. Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = 3$



Discrete version: n intervals, each of length $1/n$.

In each interval, there is a car with probability $p = 3/n$ (assume ≤ 1 car can pass by)

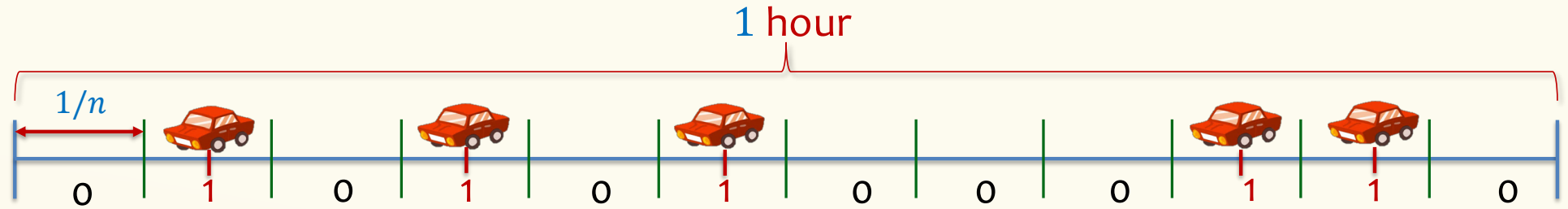
Each interval is Bernoulli: $X_i = 1$ if car in i^{th} interval (0 otherwise). $P(X_i = 1) = 3/n$

$$X = \sum_{i=1}^n X_i$$

More generally

X = # cars passing through a light in 1 hour. Disjoint time intervals are independent.

Know: $E[X] = \lambda$ for some given $\lambda > 0$



Discrete version: n intervals, each of length $1/n$.

In each interval, there is a car with probability $p = \lambda/n$ (assume ≤ 1 car can pass by)

Each interval is Bernoulli: $X_i = 1$ if car in i^{th} interval (0 otherwise). $P(X_i = 1) = \lambda/n = p$

$$X = \sum_{i=1}^n X_i$$

$$E(X) = n \cdot \frac{\lambda}{n} = \lambda.$$

$$P(X=k) = \text{Prob Bin}(n, \frac{\lambda}{n})$$

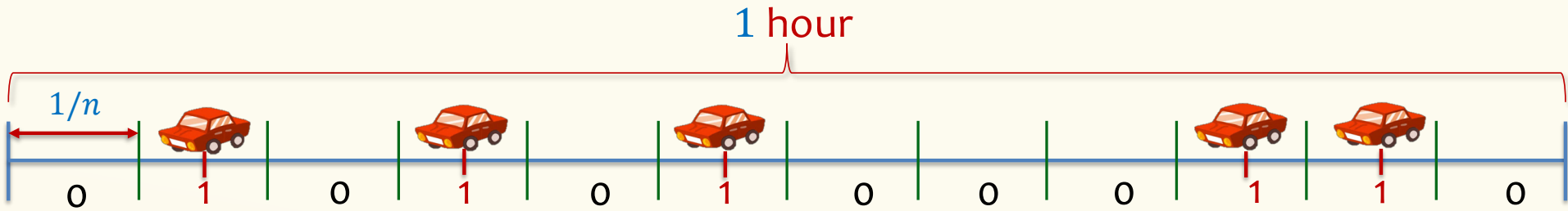
$$= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

has value k

Model as Binomial

X = # cars passing through a light in 1 hour. Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = \lambda$ for some given $\lambda > 0$



Discrete version: n intervals, each of length $1/n$.

In each interval, there is a car with probability $p = \lambda/n$ (assume ≤ 1 car can pass by)

Each interval is Bernoulli: $X_i = 1$ if car in i^{th} interval (0 otherwise). $P(X_i = 1) = \lambda/n$

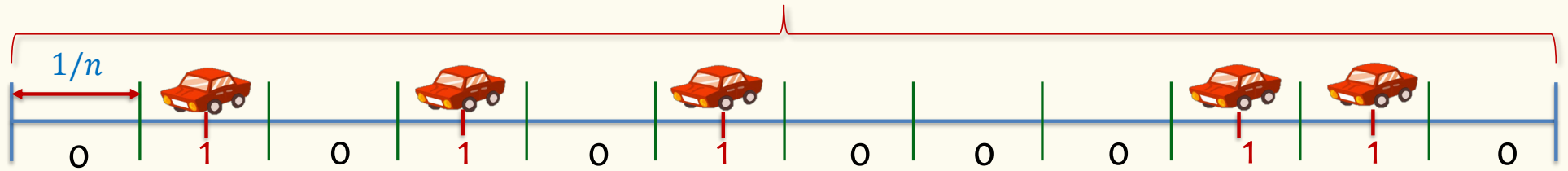
$$X = \sum_{i=1}^n X_i \quad X \sim \text{Bin}(n, p)$$

$$P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

indeed! $\mathbb{E}[X] = pn = \lambda$

Don't like discretization

$$X \text{ is binomial } P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$



We want now $n \rightarrow \infty$

$$P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

$$P(X=0) = \binom{n}{0} \left(\frac{\lambda}{n}\right)^0 \left(1 - \frac{\lambda}{n}\right)^n$$

\downarrow \downarrow
 1 1

$$\left(1 - \frac{\lambda}{n}\right)^n$$

$$\left(e^{-\frac{\lambda}{n}}\right)^n = e^{-\lambda}$$

$x \rightarrow 0$

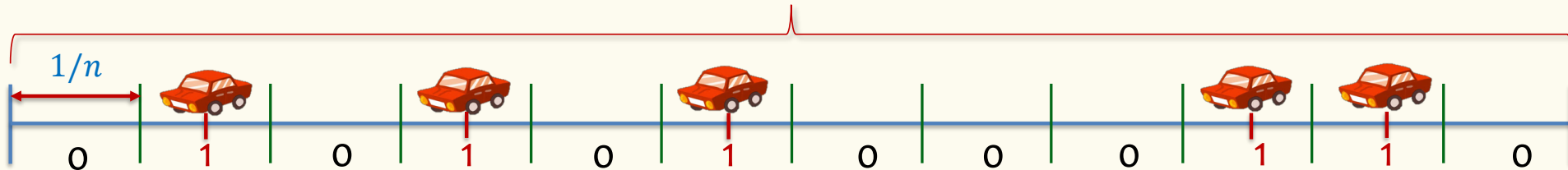
$(1-x) \approx e^{-x}$

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

Take the limit

$$X \text{ is binomial } P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$



We want now $n \rightarrow \infty$

$$\frac{n!}{(n-i)! i!}$$

$$a^x a^y = a^{x+y}$$

$$P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} = \frac{n!}{(n-i)! n^i i!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-i}$$

$\xrightarrow{1}$ $\xrightarrow{e^{-\lambda}}$ $\xrightarrow{1}$

$$\rightarrow P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Poisson Distribution

- Suppose “events” happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. with parameter λ (denoted $X \sim \text{Poi}(\lambda)$) and has distribution (PMF):

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \quad i = 0, 1, 2, \dots$$

$$\mathcal{L}_X = \{0, 1, 2, \dots\}$$

Some examples of “Poisson processes”:

- # of cars passing through a traffic light in 1 hour
- # of requests to web servers in an hour
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume
fixed average rate

Poisson Distribution

Siméon Denis Poisson
1781-1840

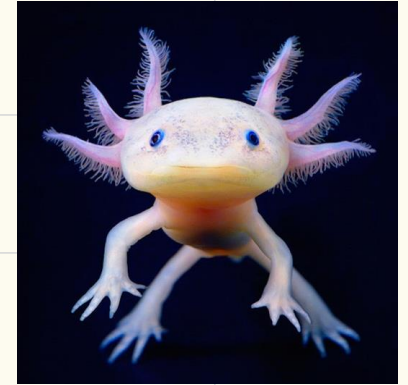
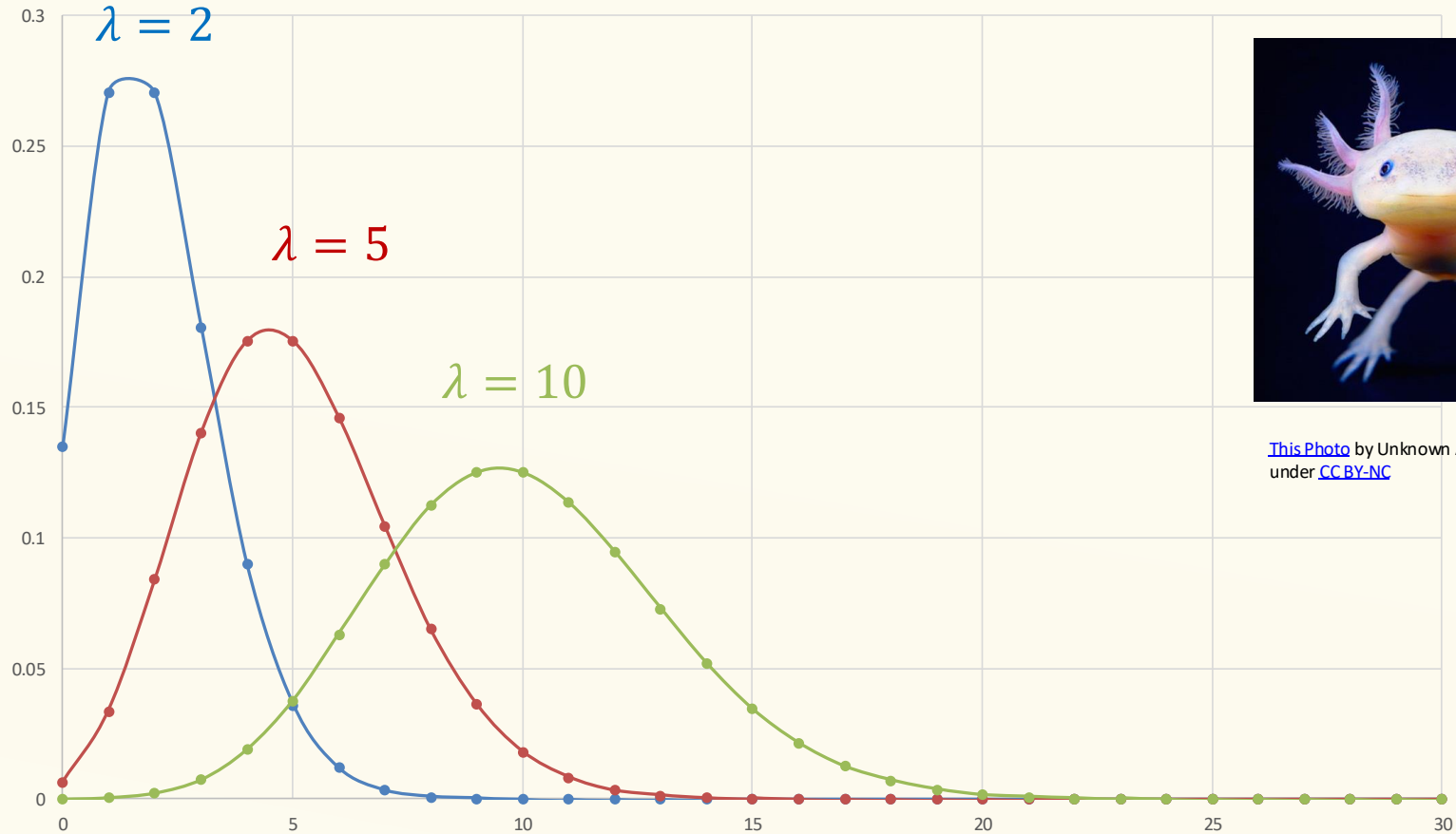
- Suppose “events” happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. with parameter λ (denoted $X \sim \text{Poi}(\lambda)$) and has distribution (PMF):

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \quad i = 0, 1, 2, \dots$$

$$\begin{aligned} E(X) &= \lambda \\ \text{Var}(X) &= \lambda \end{aligned}$$

Probability Mass Function

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



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Validity of Distribution

$$\lambda > 0$$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \quad i = 0, 1, 2, \dots$$

$$\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$$

Is this a valid probability mass function?
(How do you show that a pmf is valid?)

$$\begin{aligned} 1 &= \sum_{i=0}^{\infty} P(X=i) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} \left(\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \right) = e^{-\lambda} e^{\lambda} \\ &= e^0 = 1 \end{aligned}$$


Calculate sum

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

To show that a pmf is valid, need to check that it takes nonnegative values and that the probabilities sum up to 1.

$$\sum_{i=0}^{\infty} P(X = i) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}$$

Fact (Taylor series expansion):

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Expectation calculation

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ , then
 $\mathbb{E}[X] = \lambda$

Proof. $\mathbb{E}[X] = \sum_{i=0}^{\infty} P(X = i) \cdot i = \sum_{i=1}^{\infty} P(X=i) \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} \cdot i$

$= e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$

Expectation calculation

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ , then

$$\mathbb{E}[X] = \lambda$$

Proof.

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=0}^{\infty} P(X = i) \cdot i = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} \\ &= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \\ &= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda\end{aligned}$$

(Note: The sum $\sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!}$ is highlighted in yellow and labeled "= 1 (see prior slides!)")

Variance

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ , then $\text{Var}(X) = \lambda$

Proof.
$$\begin{aligned} \mathbb{E}[X^2] &= \sum_{i=0}^{\infty} P(X = i) \cdot i^2 = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i \\ &= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1) \\ &= \lambda \left[\underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}[X] = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda \end{aligned}$$

Similar to the previous proof
Verify offline.



$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0, 1, 2, 3 \dots$,

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



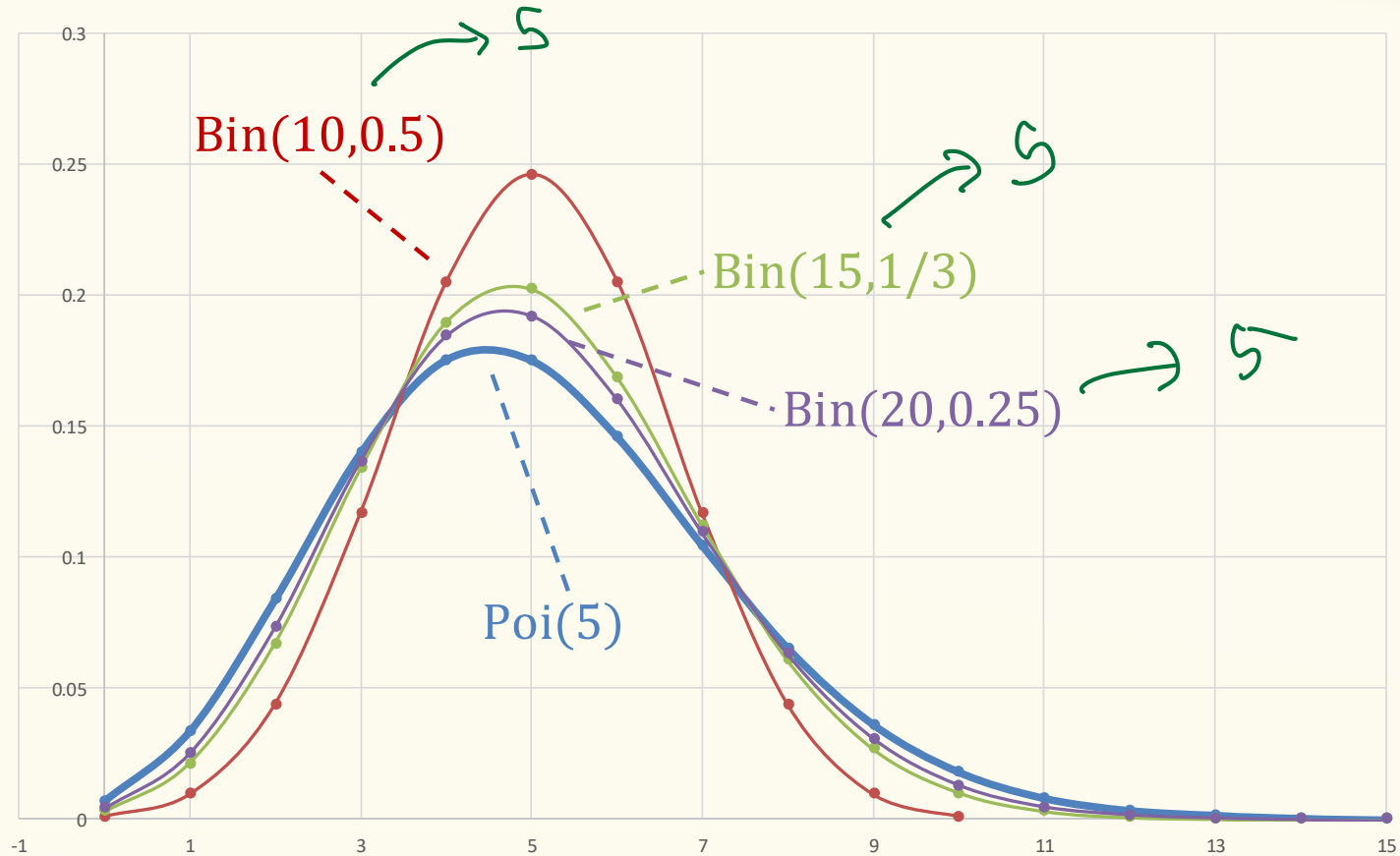
Poisson approximates Binomial when:

n is very large, p is very small, and $\lambda = np$ is “moderate”
e.g. ($n > 20$ and $p < 0.05$), ($n > 100$ and $p < 0.1$)

Formally, Binomial approaches Poisson in the limit as $n \rightarrow \infty$ (equivalently, $p \rightarrow 0$) while holding $np = \lambda$

Probability Mass Function – Convergence of Binomials

$$\lambda = 5$$
$$p = \frac{5}{n}$$
$$n = 10, 15, 20$$



as $n \rightarrow \infty$, $\text{Binomial}(n, p = \lambda/n) \rightarrow \text{poi}(\lambda)$

From Binomial to Poisson

$$X \sim \text{Bin}(n, p)$$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np$$

$$\text{Var}(X) = np(1 - p)$$

$$n \rightarrow \infty$$

$$np = \lambda$$

$$p = \frac{\lambda}{n} \rightarrow 0$$



$$X \sim \text{Poi}(\lambda)$$

$$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$E[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

Example -- Approximate Binomial Using Poisson

Consider sending bit string over a network

- Send bit string of length $n = 10^4$
- Probability of (independent) bit corruption is $p = 10^{-6}$

$$Y = \# \text{ corrupted bits} \\ \sim \text{Bin}(10^4, 10^{-6})$$

What is probability that message arrives uncorrupted?

Using $X \sim \text{Poi}(\lambda = np = 10^4 \cdot 10^{-6} = 0.01)$

$$P(X = 0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-0.01} \cdot \frac{0.01^0}{0!} \approx 0.990049834$$

Using $Y \sim \text{Bin}(10^4, 10^{-6})$

$$P(Y = 0) \approx 0.990049829$$

Sum of Independent Poisson RVs

Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

X, Y indep.

Let $Z = X + Y$. What kind of random variable is Z ?

Aka what is the “distribution” of Z ?

Intuition first:

- X is measuring number of (type 1) events that happen in, say, an hour if they happen at an average rate of λ_1 per hour.
- Y is measuring number of (type 2) events that happen in, say, an hour if they happen at an average rate of λ_2 per hour.
- Z is measuring total number of events of both types that happen in, say, an hour, if type 1 and type 2 events occur independently.

Sum of Independent Poisson RVs (2)

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ be independent r.v.s such that $\lambda = \lambda_1 + \lambda_2$. Let $Z = X + Y$. For all $z = 0, 1, 2, 3, \dots$,

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

$$Z \sim \text{Poi}(\lambda_1 + \lambda_2)$$

More generally, let $X_1 \sim \text{Poi}(\lambda_1), \dots, X_n \sim \text{Poi}(\lambda_n)$ independent such that $\lambda = \sum_i \lambda_i$. Let $Z = \sum_i X_i$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

$$Z \sim \text{Poi}(\underbrace{\lambda_1 + \lambda_2 + \dots + \lambda_n}_{\lambda})$$

Sum of Independent Poisson RVs (3)

X	0	1	2	...	z
Y	z	z-1	z-2		0
					$\frac{z+1}{z}$
					-1

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = X + Y$. For all $z = 0, 1, 2, 3, \dots$,

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

$P((X+Y)=z)$
 $P(Z = z) = ?$

$P(A \text{ and } B) = P(B|A)P(A)$

- $P(Z = z) = \sum_{j=0}^z P(X = j, Y = z - j)$
- $P(Z = z) = \sum_{j=0}^{\infty} P(X = j, Y = z - j)$
- $P(Z = z) = \sum_{j=0}^z P(Y = z - j | X = j) P(X = j)$
- $P(Z = z) = \sum_{j=0}^z P(Y = z - j | X = j)$

<https://pollev.com/annakarlin185>

- All of them are right
- The first 3 are right
- Only 1 is right
- Don't know

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = X + Y$. For all $z = 0, 1, 2, 3, \dots$,

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

Proof

$$P(Z = z) = \sum_{j=0}^z P(X = j, Y = z - j) \quad \text{Law of total probability}$$

Proof

$$P(Z = z) = \sum_{j=0}^z P(X = j, Y = z - j)$$

Law of total probability

$$= \sum_{j=0}^z P(X = j) P(Y = z - j) = \sum_{j=0}^z e^{-\lambda_1} \cdot \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{z-j}}{z-j!}$$

Independence

$$= e^{-\lambda_1 - \lambda_2} \left(\sum_{j=0}^z \frac{1}{j! z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right)$$

$$= e^{-\lambda} \left(\sum_{j=0}^z \frac{z!}{j! z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right) \frac{1}{z!}$$

$$= e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^z \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^z \cdot \frac{1}{z!}$$

Binomial
Theorem

Poisson Random Variables - Summary

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0, 1, 2, 3 \dots$,

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

General principle:

- Events happen at an average rate of λ per time unit
- Number of events happening at a time unit X is distributed according to $\text{Poi}(\lambda)$
- Poisson approximates Binomial when n is large, p is small, and np is moderate
- Sum of independent Poisson is still a Poisson

Who Fails First?

You have a disk with probability p_1 of failing each day, and a CPU which independently has probability p_2 of failing each day.

$$P\{\text{disk fails before CPU fails}\} = \frac{p_1(1-p_2)}{1-(1-p_2)(1-p_1)}$$

Intuition: Think about flipping 2 coins each day.

There may be many days where both coins are heads.

We only care about the *first day where the coins are not both heads*.

Given that both coins are not heads, what's the probability that coin 1 is H and coin 2 is T?

$$P\{\text{coin 1 is H \& coin 2 is T} \mid \text{not both tails}\} = \frac{P\{\text{coin 1 is H \& coin 2 is T}\}}{P\{\text{not both tails}\}} = \frac{p_1(1-p_2)}{1-(1-p_2)(1-p_1)}$$