Zoo of random variables

CSE 312 Spring 26

Lecture 12

Variance - summary

Definition. The **variance** of a (discrete) RV X is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \Omega X} p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Recall $\mathbb{E}[X]$ is a **constant**, not a random variable itself.

Standard deviation: $\sigma(X) = \sqrt{\text{Var}(X)}$

Intuition: Variance (or standard deviation) is a quantity that measures, in expectation, how "far" the random variable is from its expectation.

Variance – Properties (3)

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$$

Definition. The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \Omega_X} p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Theorem. $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Theorem. For any $a, b \in \mathbb{R}$, $\mathrm{Var}(a \cdot X + b) = a^2 \cdot \mathrm{Var}(X)$

Multiple Random Variables and Independence

Comma is shorthand for AND

Definition. Two random variables X, Y are **(mutually) independent** if for all x, y,

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$$

Intuition: Knowing X doesn't help you guess Y and vice versa

Definition. The random variables $X_1, \ldots, X_n$ are **(mutually) independent** if for all $x_1, \ldots, x_n$,

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdots P(X_n = x_n)$$

Note: No need to check for all subsets, but need to check for all outcomes!

Important Facts about Independent Random Variables

Theorem. If X, Y independent, $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Theorem. If X, Y independent, $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Corollary. If $X_1, X_2, \dots, X_n$ mutually independent,

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i}^{n} \mathrm{Var}(X_i)$$

Agenda

- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Motivation for “Named” Random Variables

Random Variables that show up all over the place.

- Easily solve a problem by recognizing it’s a special case of one of these random variables.

Each RV introduced today will show:

- A general situation it models
- Its name and parameters
- Its PMF, Expectation, and Variance
- Example scenarios you can use it

Welcome to the Zoo! (Preview) 🦍🐘🦁🐅🦓🐪🦒

$X \sim \mathrm{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$
$$\mathbb{E}[X] = \frac{a + b}{2}$$
$$\mathrm{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \mathrm{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$
$$\mathbb{E}[X] = p$$
$$\mathrm{Var}(X) = p(1 - p)$$

$X \sim \mathrm{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$
$$\mathbb{E}[X] = np$$
$$\mathrm{Var}(X) = np(1 - p)$$

$X \sim \mathrm{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$
$$\mathbb{E}[X] = \frac{1}{p}$$
$$\mathrm{Var}(X) = \frac{1 - p}{p^2}$$

$X \sim \mathrm{NegBin}(r, p)$

$$P(X = k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$
$$\mathbb{E}[X] = \frac{r}{p}$$
$$\mathrm{Var}(X) = \frac{r(1 - p)}{p^2}$$

$X \sim \mathrm{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$
$$\mathbb{E}[X] = n\frac{K}{N}$$
$$\mathrm{Var}(X) = n\frac{K(N - K)(N - n)}{N^2(N - 1)}$$

Zoo of Discrete RVs, Part I

- Uniform Random Variables
- Bernoulli Random Variables
- Binomial Random Variables
- Geometric Random Variables

Discrete Uniform Random Variables

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is uniform.

Notation:

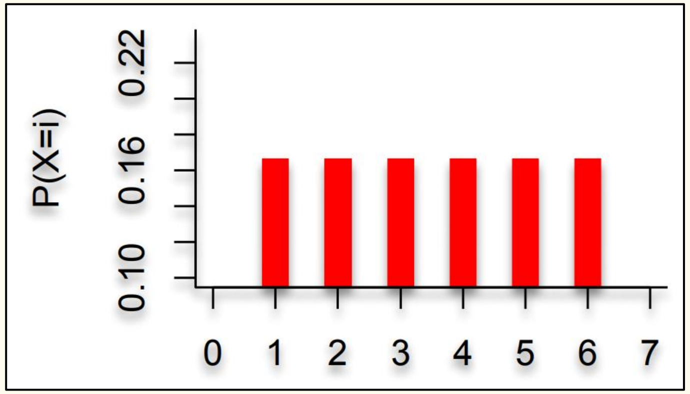
PMF:

Expectation:

Variance:

Example: value shown on one roll of a fair die is $\mathrm{Unif}(1{,}6)$:

- $P(X = i) = 1/6$
- $\mathbb{E}[X] = 7/2$
- $\mathrm{Var}(X) = 35/12$



Discrete Uniform Random Variables (2)

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is uniform.

Notation: $X \sim \mathrm{Unif}(a, b)$

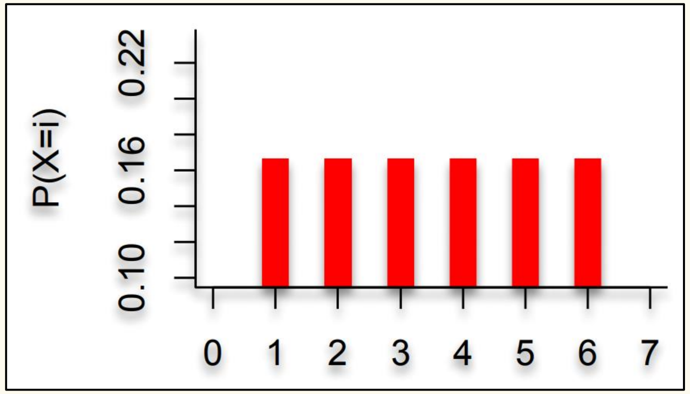
PMF: $\mathrm{P}(X = i) = \frac{1}{b - a + 1}$

Expectation: $\mathbb{E}[X] = \frac{a+b}{2}$

Variance: $\mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$

Example: value shown on one roll of a fair die is $\mathrm{Unif}(1,6)$:

- $P(X = i) = 1/6$
- $\mathbb{E}[X] = 7/2$
- $\mathrm{Var}(X) = 35/12$



Zoo (2)

- Zoo of Discrete RVs
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Bernoulli Random Variables

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \mathrm{Ber}(p)$

PMF: $P(X = 1) = p,\ P(X = 0) = 1 - p$

Expectation:

Variance:

Poll:
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	Mean	Variance
A.	p	p
B.	p	$1 - p$
C.	p	$p(1 - p)$
D.	p	p^2

Bernoulli Random Variables (2)

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \mathrm{Ber}(p)$

PMF: $P(X = 1) = p,\ P(X = 0) = 1 - p$

Expectation: $\mathbb{E}[X] = p$ Note: $\mathbb{E}[X^2] = p$

Variance: $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = p(1 - p)$

Examples:

- Coin flip
- Randomly guessing on a MC test question
- A server in a cluster fails
- Whether or not a share of a particular stock pays off or not
- Any indicator r.v.

Zoo (3)

- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Binomial Random Variables

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \text{Ber}(p)$.

Counts number of successes in n independent trials, each with probability p of success.

X is a Binomial random variable

Examples:

- # of heads in n indep coin flips
- # of 1s in a randomly generated n bit string
- # of servers that fail in a cluster of n computers
- # of bit errors in file written to disk
- # of elements in a bucket of a large hash table
- # of n different stocks that "pay off"

Poll:
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$\boldsymbol{P(X = k) =}$

A. $p^k(1-p)^{n-k}$
B. np
C. $\binom{n}{k} p^k(1-p)^{n-k}$
D. $\binom{n}{n-k} p^k(1-p)^{n-k}$

Binomial Random Variables (2)

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \mathrm{Ber}(p)$.
Counts number of successes in n independent trials, each with probability p of success.

X is a Binomial random variable

Poll:
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	Mean	Variance
A.	p	p
B.	np	$np(1-p)$
C.	np	np^2
D.	np	n^2p

Notation: $X \sim \mathrm{Bin}(n, p)$

PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation:

Variance:

Binomial Random Variables (3)

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \text{Ber}(p)$.

Counts number of successes in n independent trials, each with probability p of success.

X is a Binomial random variable

Notation: $X \sim \text{Bin}(n, p)$

PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation: $\mathbb{E}[X] = np$

Variance: $\text{Var}(X) = np(1-p)$

Mean, Variance of the Binomial

"i.i.d." is a commonly used phrase.
It means "independent & identically distributed"

If $Y_1, Y_2, \dots, Y_n \sim \text{Ber}(p)$ and independent (i.i.d.), then
$X = \sum_{i=1}^{n} Y_i, \quad X \sim \text{Bin}(n, p)$

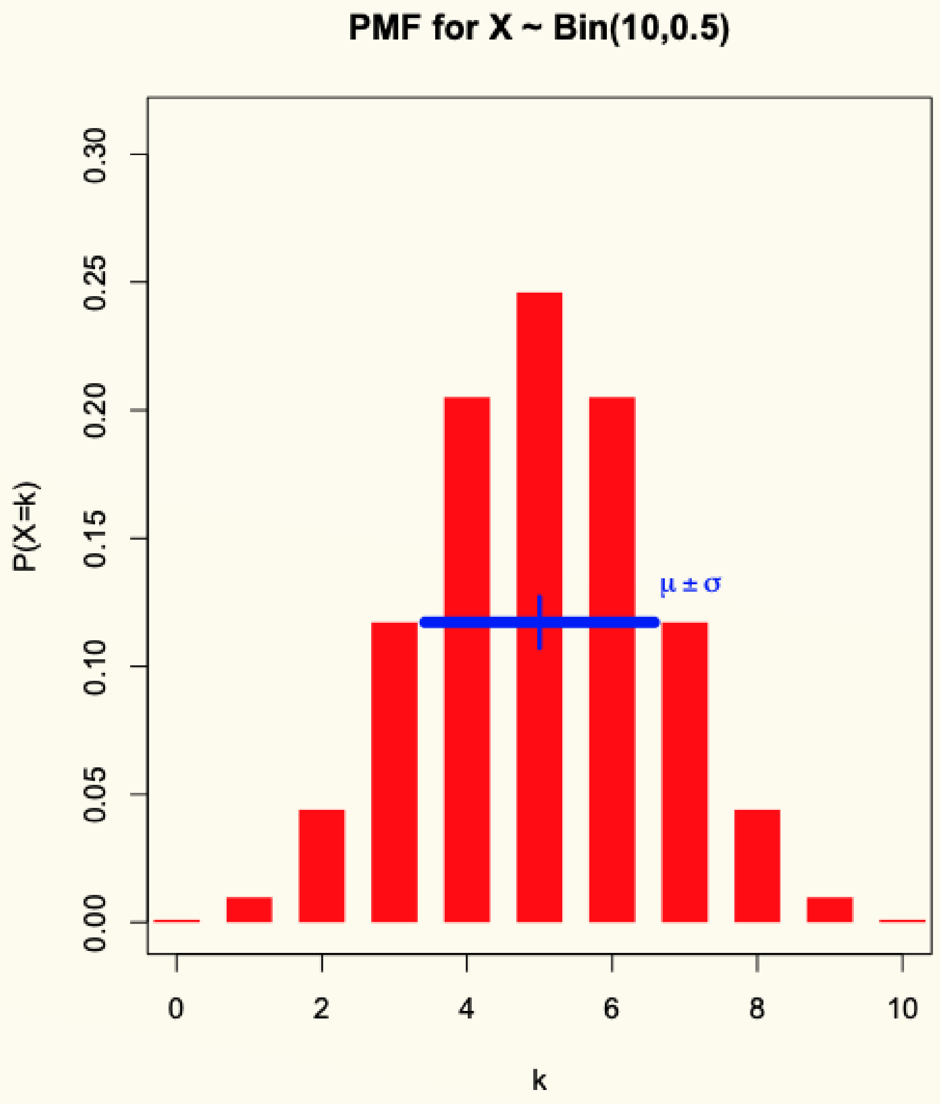
Claim $\mathbb{E}[X] = np$

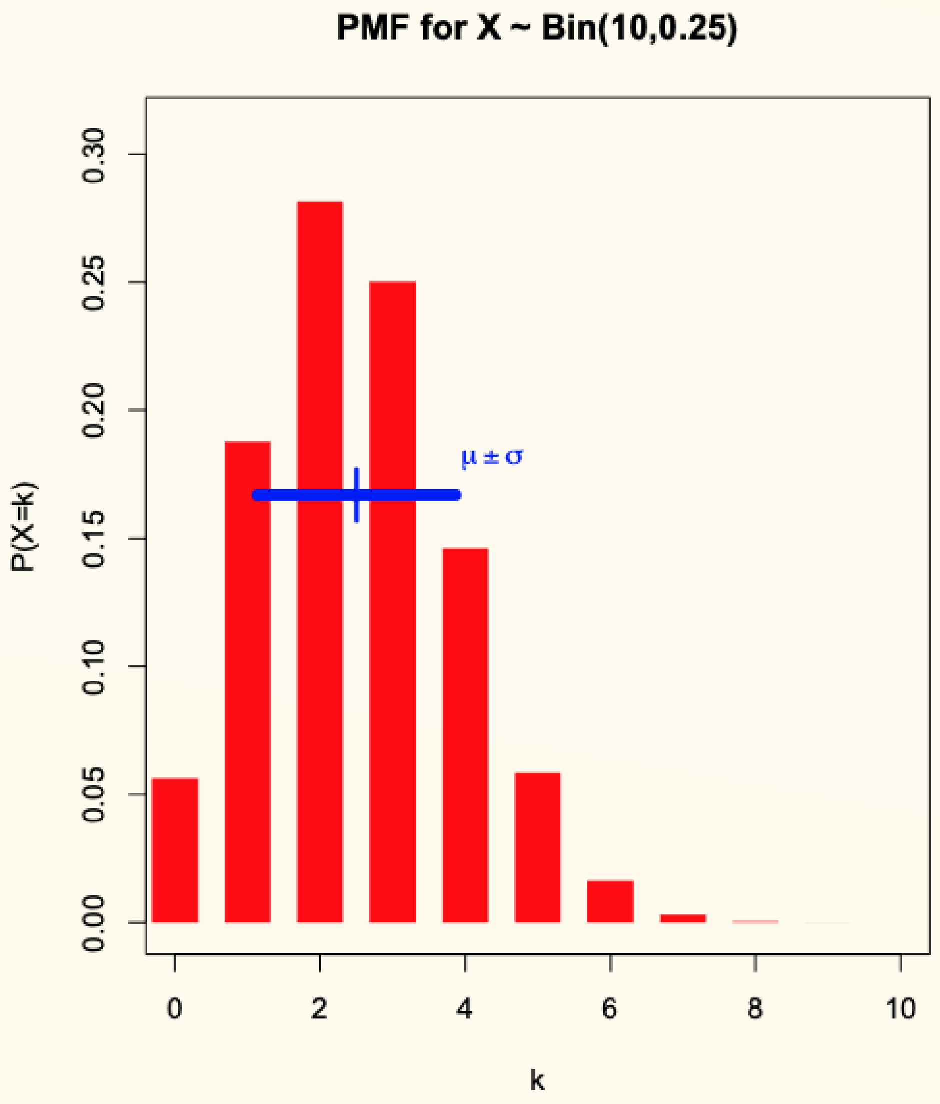
$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = n\mathbb{E}[Y_1] = np$$

Claim $\text{Var}(X) = np(1-p)$, by independence of r.v.'s

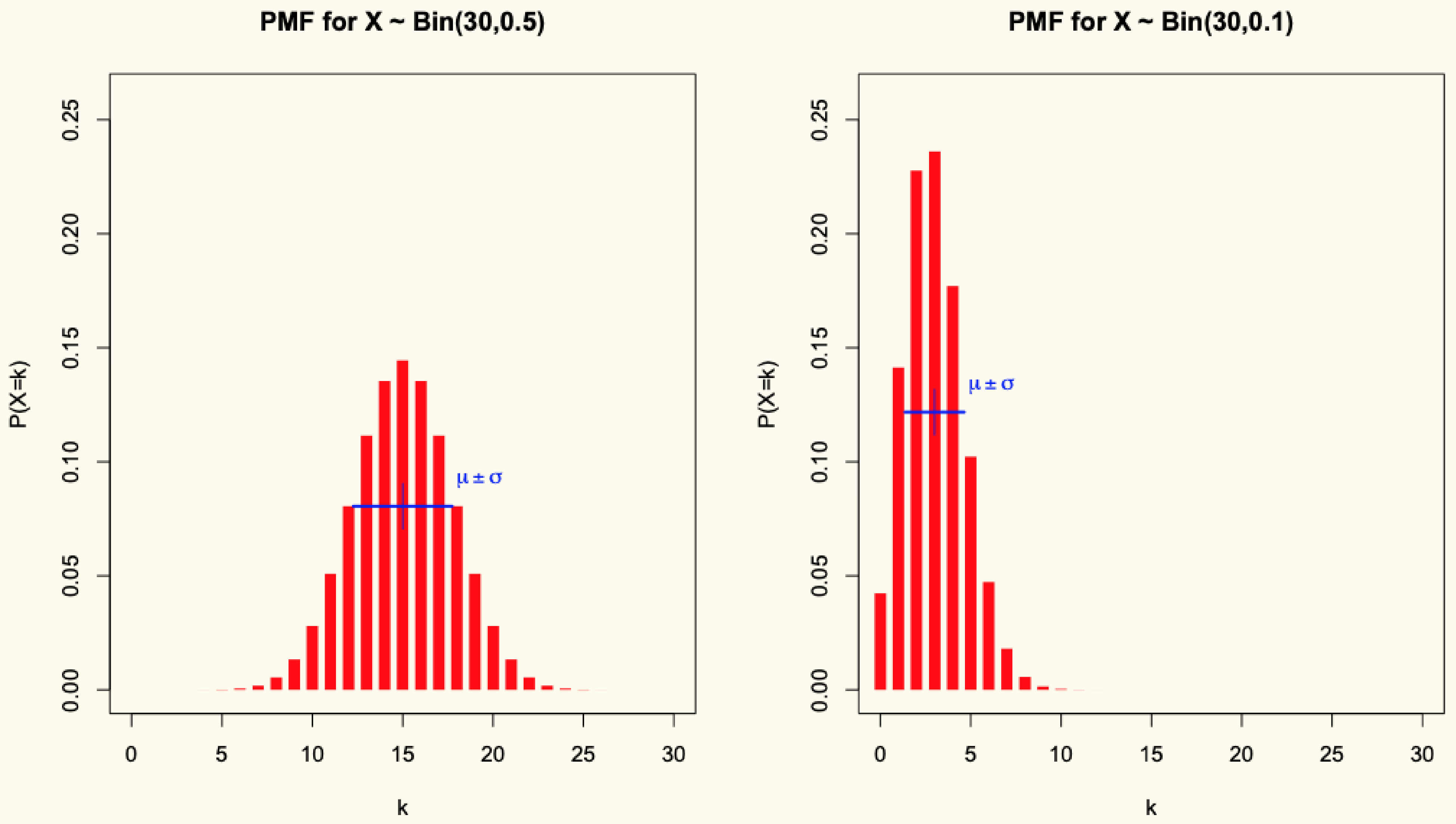
$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \text{Var}(Y_i) = n\text{Var}(Y_1) = np(1-p)$$

Binomial PMFs





Binomial PMFs (2)



Zoo (4)

- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Geometric Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim \mathrm{Ber}(p)$ up until and including the first success.

X is called a Geometric random variable with parameter p.

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

Notation: $X \sim \mathrm{Geo}(p)$

PMF:

Expectation:

Variance:

Geometric Random Variables (2)

A discrete random variable X that models the number of independent trials $Y_i \sim \text{Ber}(p)$ up until and including the first success.

X is called a Geometric random variable with parameter p.

Notation: $X \sim \text{Geo}(p)$

PMF: $P(X = k) = (1 - p)^{k-1}p$

Expectation: $\mathbb{E}[X] = \frac{1}{p}$

Variance: $\text{Var}(X) = \frac{1-p}{p^2}$

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it
- # hash trials until a miner successfully mines a Bitcoin

Zoo (5)

- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables
  - More examples

Example 1

Sending a binary message of length 1024 bits over a network with probability 0.999 of correctly sending each bit in the message without corruption (independent of other bits).

Let X be the number of corrupted bits.

What kind of random variable is this and what is $\mathbb{E}[X]$?

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A 1022.99
B 1.024
C 1.02298
D. 1

Example

Sending a binary message of length 1024 bits over a network with probability 0.999 of correctly sending each bit in the message without corruption (independent of other bits).

Let X be the number of corrupted bits.

What kind of random variable is this and what is $\mathbb{E}[X]$?

Binomial (1024, 0.001)

Therefore $\mathbb{E}[X] = np = 1024 \cdot 0.001 = 1.024$

Example: Music Lessons

Your music teacher requires you to play a 1000 note song without mistake. You have been practicing, so you have a probability of 0.999 of getting each note correct (independent of the others). If you mess up a single note in the song, you must start over and play from the beginning. Let X be the number of times you have to play the song from the start. What kind of random variable is this and what is $\mathbb{E}[X]$?

Example: Music Lessons - solution

Your music teacher requires you to play a 1000 note song without mistake. You have been practicing, so you have a probability of 0.999 of getting each note correct (independent of the others). If you mess up a single note in the song, you must start over and play from the beginning. Let X be the number of times you have to play the song from the start. What kind of random variable is this and what is $\mathbb{E}[X]$?

Probability that you play whole song without a mistake is 0.999^{1000}

Therefore X is a Geometric random variable with parameter $p = 0.999^{1000}$

So its expectation is $\frac{1}{0.999^{1000}}$

Zoo (6)

- Zoo of Discrete RVs
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random variables
  - Poisson Distribution

Preview: Poisson

Model: X is # events that occur in an hour

- Expect to see 3 events per hour (but will be random)
- The expected number of events in t hours, is $3t$
- Occurrence of events on disjoint time intervals is independent

Example – Modelling car arrivals at an intersection

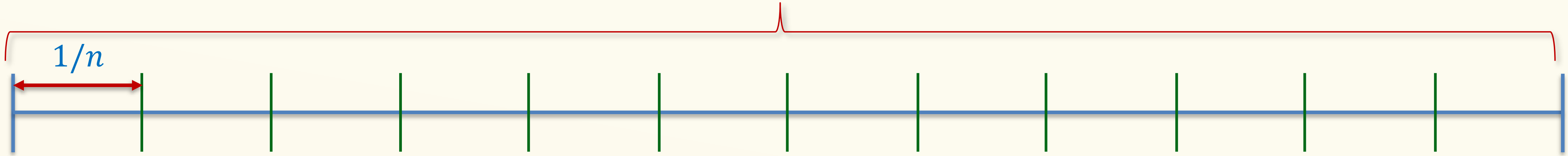
X = # of cars passing through a light in 1 hour

Example – Model the process of cars passing through a light in 1 hour

X = # cars passing through a light in 1 hour. $\mathbb{E}[X] = 3$

Assume: Occurrence of events on disjoint time intervals is independent

Approximation idea: Divide hour into n intervals of length $1/n$



This gives us n independent intervals

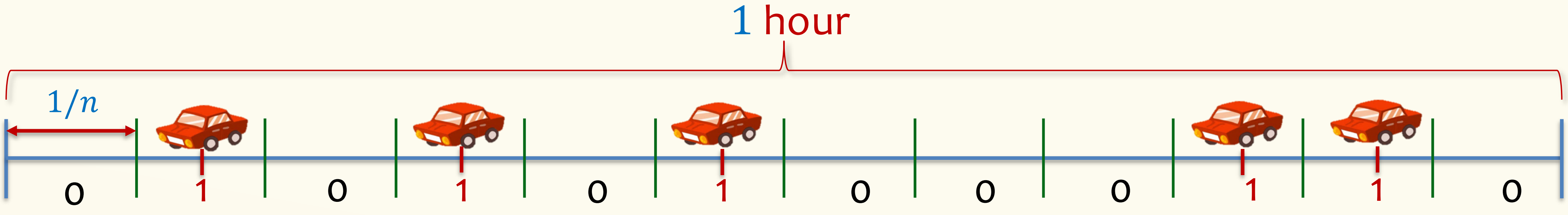
Assume either zero or one car per interval

$p =$ probability car arrives in a single interval of length $1/n$

What should p be?

X = # cars passing through a light in 1 hour.   Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = 3$



This gives us n independent intervals

Assume either zero or one car per interval

$p =$ probability car arrives in an interval

Model as $\text{Bin}(n, p)$

What should p be?

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A. $3/n$

B. $3n$

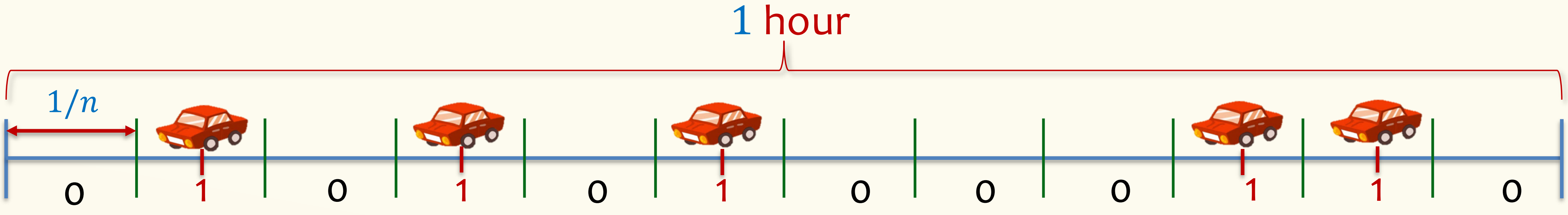
C. 3

D. $3/60$

Model as Binomial

X = # cars passing through a light in 1 hour. Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = \lambda$ for some given $\lambda > 0$



Discrete version: n intervals, each of length $1/n$.

In each interval, there is a car with probability $p = \lambda/n$ (assume ≤ 1 car can pass by)

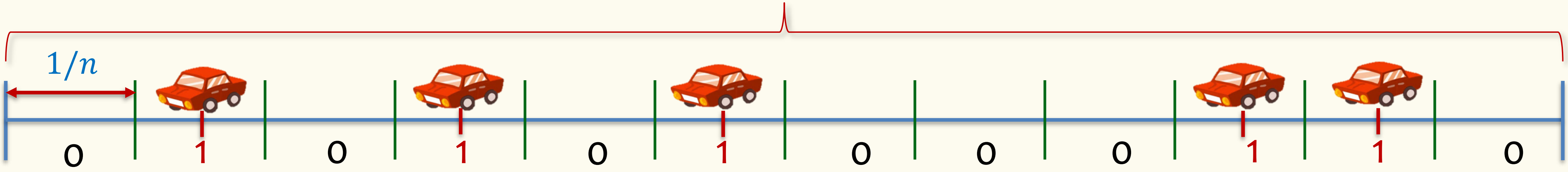
Each interval is Bernoulli: $X_i = 1$ if car in i^{th} interval (0 otherwise). $P(X_i = 1) = \lambda / n$

$X = \sum_{i=1}^{n} X_i$ $\quad$ $X \sim \text{Bin}(n, p)$ $\quad$ $P(X = i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1 - \frac{\lambda}{n}\right)^{n-i}$

indeed! $\mathbb{E}[X] = pn = \lambda$

Don’t like discretization

X is binomial $P(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}$

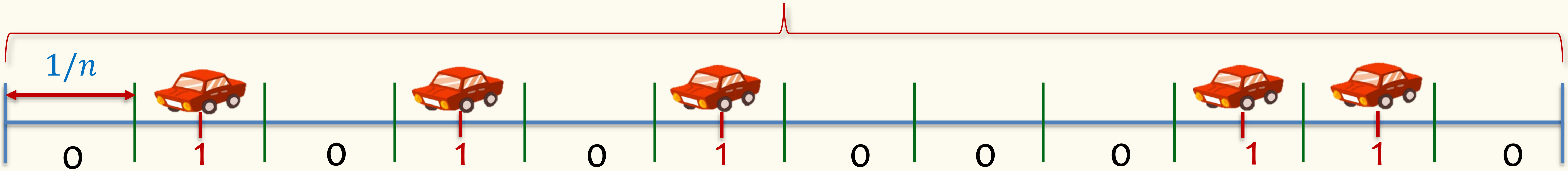


We want now $n \to \infty$

$$P(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}$$

Take the limit

X is binomial $P(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}$



We want now $n \to \infty$

$$P(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i} = \underbrace{\frac{n!}{(n-i)!\,n^i}}_{\to 1}\frac{\lambda^i}{i!}\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}}\underbrace{\left(1-\frac{\lambda}{n}\right)^{-i}}_{\to 1}$$

$$\to P(X=i)=e^{-\lambda}\cdot\frac{\lambda^i}{i!}$$

Poisson Distribution

Siméon Denis Poisson
1781-1840

- Suppose "events" happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. *with parameter* λ (denoted $X \sim \mathrm{Poi}(\lambda)$) and has distribution (PMF):

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

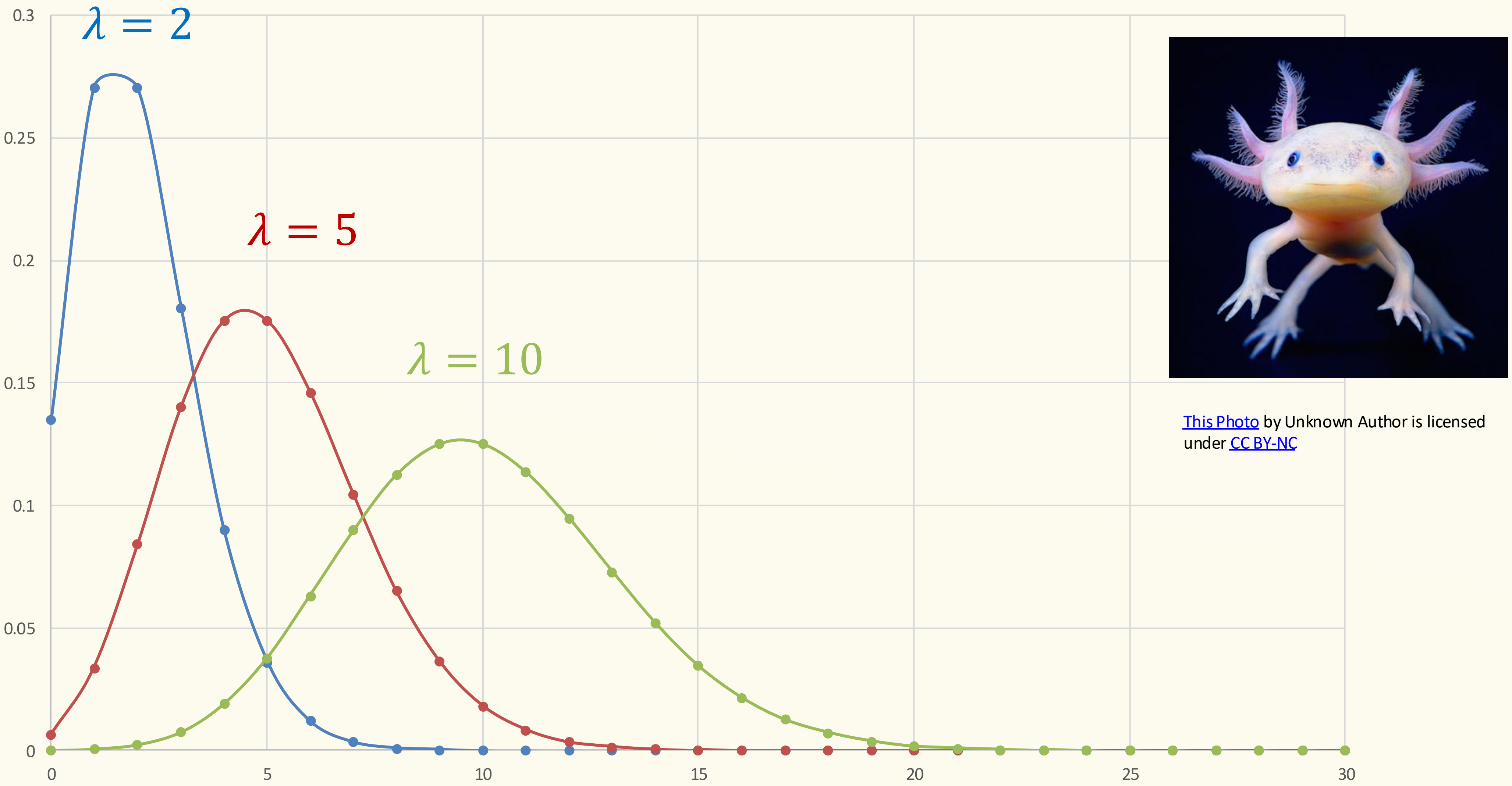
Several examples of "Poisson processes":

- # of cars passing through a traffic light in 1 hour
- # of requests to web servers in an hour
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume fixed average rate

Probability Mass Function

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



Validity of Distribution

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

Is this a valid probability mass function?
(How do you show that a pmf is valid?)

Calculate sum

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

To show that a pmf is valid, need to check that it takes nonnegative values and that the probabilities sum up to 1.

$$\sum_{i=0}^{\infty} P(X = i) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}$$

Fact (Taylor series expansion):

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Expectation

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter $\lambda \geq 0$, then

$$\mathbb{E}[X] = ?$$

Proof. $$\mathbb{E}[X] = \sum_{i=0}^{\infty} P(X = i) \cdot i = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i$$

Expectation calculation

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then

$$\mathbb{E}[X] = \lambda$$

Proof.

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} P(X = i) \cdot i = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

= 1 (see prior slides!)

$$= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda$$

Variance

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then $\mathrm{Var}(X) = \lambda$

Proof. $$\mathbb{E}[X^2] = \sum_{i=0}^{\infty} P(X = i) \cdot i^2 = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1)$$

$$= \lambda \left[\underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}[X] = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda$$

Similar to the previous proof
Verify offline.

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$