Variance, Independent Random Variables, Begin Zoo

CSE 312 Spring 26

Lecture 11

Review Expected Value of a Random Variable

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x) \qquad = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Recap Linearity of Expectation

Theorem. For any two random variables X and Y

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Or, more generally: For any random variables $X_1, \ldots, X_n$, and constants $a_1, a_2, \ldots, a_n, b$,

$$\mathbb{E}[a_1 X_1 + \cdots + a_n X_n + b] = a_1 \mathbb{E}[X_1] + \cdots + an\mathbb{E}[X_n] + b.$$

Recap Using LOE to compute complicated expectations

Often boils down to the following three steps:

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

- Conquer: Compute the expectation of each X_i

Often, X_i are **indicator** (0/1) random variables.

Recap Expected Value of $g(X)$ -- LOTUS

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x) \quad = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

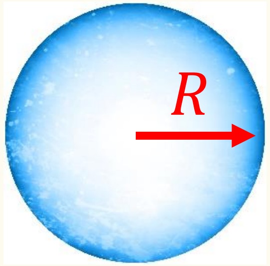
Also known as **LOTUS:** "Law of the unconscious statistician

LOTUS Example

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x) \quad = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Consider a sphere, whose radius is a random variable R:



$$R = \begin{cases} 1 & \text{w.p. } \frac{3}{8} \\ 2 & \text{w.p. } \frac{1}{4} \\ 3 & \text{w.p. } \frac{3}{8} \end{cases}$$

Q: What is the expected radius of the sphere?

Q: What is the expected volume of the sphere?

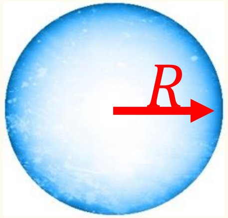
LOTUS example (2)

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x) = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Consider a sphere, whose radius is a random variable R:

$$R = \begin{cases} 1 & \text{w.p. } \frac{3}{8} \\ 2 & \text{w.p. } \frac{1}{4} \\ 3 & \text{w.p. } \frac{3}{8} \end{cases}$$



$$\mathbb{E}[\text{Volume}] = \mathbb{E}\left[\frac{4}{3}\pi R^3\right]$$

Q: Is $\mathbb{E}[R^3] = (\mathbb{E}[R]^3)$?

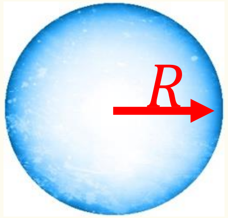
LOTUS example (3)

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x) \quad = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Consider a sphere, whose radius is a random variable R:

$$R = \begin{cases} 1 & \text{w.p. } \frac{3}{8} \\ 2 & \text{w.p. } \frac{1}{4} \\ 3 & \text{w.p. } \frac{3}{8} \end{cases}$$



$$\mathbb{E}[\text{Volume}] = \mathbb{E}\left[\frac{4}{3}\pi R^3\right]$$

$$= \frac{4}{3}\pi \cdot 1^3 \cdot \frac{3}{8} + \frac{4}{3}\pi \cdot 2^3 \cdot \frac{1}{4} + \frac{4}{3}\pi \cdot 3^3 \cdot \frac{3}{8}$$

$$= 7\frac{2}{3}\pi$$

Variance – Properties

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$$

Definition. The **variance** of a (discrete) RV X is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \Omega_X} p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Intuition: Variance (or standard deviation) is a quantity that measures, in expectation, how "far" the random variable is from its expectation.

Variance - summary

Definition. The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \Omega X} p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Recall $\mathbb{E}[X]$ is a **constant**, not a random variable itself.

Standard deviation: $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

Intuition: Variance (or standard deviation) is a quantity that measures, in expectation, how "far" the random variable is from its expectation.

Agenda

- Recap
- Properties of Variance
- Independent Random Variables
- Properties of Independent Random Variables

Variance – Properties (1)

Definition. The **variance** of a (discrete) RV X is

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_x p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Theorem. $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Variance – Properties (2)

Theorem. $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Proof: $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$

Recall $\mathbb{E}[X]$ is a **constant**

$$= \mathbb{E}[X^2 - 2\mathbb{E}[X] \cdot X + \mathbb{E}[X]^2]$$

$$= \mathbb{E}(X^2) - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2$$

(linearity of expectation!)

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$\mathbb{E}[X^2]$ and $\mathbb{E}[X]^2$ are different !

Variance – Properties (3)

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$$

Definition. The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \Omega_X} p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Theorem. $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Theorem. For any $a, b \in \mathbb{R}$, $\mathrm{Var}(a \cdot X + b) = a^2 \cdot \mathrm{Var}(X)$

Variance of Indicator Random Variables

Suppose that X_A is an indicator RV for event A with $P(A) = p$ so

$$\mathbb{E}[X_A] = P(A) = p$$

$$\text{Var}(X_A) = \mathbb{E}[X_A^2] - \mathbb{E}[X_A]^2 =$$

Variance of Indicator Random Variables - calculation

Suppose that X_A is an indicator RV for event A with $P(A) = p$ so

$$\mathbb{E}[X_A] = P(A) = p$$

Since X_A only takes on values 0 and 1, we always have $X_A^2 = X_A$ so

$$\mathrm{Var}(X_A) = \mathbb{E}[X_A^2] - \mathbb{E}[X_A]^2 = \mathbb{E}[X_A] - \mathbb{E}[X_A]^2 = p - p^2 = p(1-p)$$

Another example: $\mathrm{Var}(X+Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y)$

Proof by counter-example:

Recall glued coins

- Let X_1 be a r.v. that indicates if the first coin comes up heads.
- Let X_2 be a r.v. that indicates if the second coin comes up heads.

Another example: $\mathrm{Var}(X+Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y)$ (Explanation)

Proof by counter-example:

Recall glued coins

- Let X_1 be a r.v. that indicates if the first coin comes up heads.
- Let X_2 be a r.v. that indicates if the second coin comes up heads.

- Outcomes are HT and TH, each with probability 0.5
- Therefore, X_1 and X_2 are indicator random variables with probability 0.5 of being 1.
- Therefore, they both have expectation 0.5 and variance 0.25.
- Thus $\mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 0.5$
- On the other hand, $X_1 + X_2$ counts the number of heads in the outcome, which is always 1. Therefore $\mathrm{Var}(X_1 + X_2) = 0$

Question 1

The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_x p_X(x) \cdot (x - \mathbb{E}[X])^2.$$

Can the variance of a random variable be negative?

Poll:
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A. Yes
B. No
C. It will take me too long to figure out.
D. I don't know how/where to start.

Question 2

The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_x p_X(x) \cdot (x - \mathbb{E}[X])^2.$$

Is Var(X + 5) = Var (X) + 5?

Poll:

https://pollev.com/ annakarlin185

A. Yes
B. No
C. It will take me too long to figure out.
D. I don't know how/where to start.

Question 3

The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_x p_X(x) \cdot (x - \mathbb{E}[X])^2.$$

Is it true that if Var(X) = 0, then X is a constant?

Poll:

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A. Yes
B. No
C. It will take me too long to figure out.
D. I don't know how/where to start.

Question 4

The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

What is the relationship between $E(X^2)$ and $[E(X)]^2$?

Poll:

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A. $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$ for all X
B. $\mathbb{E}[X^2] \leq \mathbb{E}[X]^2$ for all X
C. $\mathbb{E}[X^2] = \mathbb{E}[X]^2$ for all X
D. None of the above.

Agenda (2)

- Variance
- Properties of Variance
- Independent Random Variables
- Properties of Independent Random Variables

Random Variables and Independence

Comma is shorthand for AND

Definition. Two random variables X, Y are **(mutually) independent** if for all x, y,

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$$

Intuition: Knowing X doesn't help you guess Y and vice versa

Multiple Random Variables and Independence

Comma is shorthand for AND

Definition. Two random variables X, Y are **(mutually) independent** if for all x, y,

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$$

Intuition: Knowing X doesn't help you guess Y and vice versa

Definition. The random variables $X_1, \ldots, X_n$ are **(mutually) independent** if for all $x_1, \ldots, x_n$,

$$P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdots P(X_n = x_n)$$

Note: No need to check for all subsets, but need to check <u>for all outcomes</u>!

These random variables are not independent

Recall glued coins

- Let X_1 be a r.v. that indicates if the first coin comes up heads.
- Let X_2 be a r.v. that indicates if the second coin comes up heads.

Example

Let X be the number of heads in n *independent* coin flips of the same coin. Let $Y = X \bmod 2$ be the parity (even/odd) of X.

Are X and Y independent?

Poll:

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A. Yes
B. No
C. It will take me too long to figure out.
D. I don't know how to start.

Example 2

Make $2n$ *independent* coin flips of the same coin.

Let X be the number of heads in the first n flips and Y be the number of heads in the last n flips.

Are X and Y independent?

Poll:
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A. Yes
B. No
C. It will take me too long to figure out.
D. I don't know how to start.

Agenda (3)

- Variance
- Properties of Variance
- Independent Random Variables
- Properties of Independent Random Variables

Important Facts about Independent Random Variables

Theorem. If X, Y independent, $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Theorem. If X, Y independent, $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Corollary. If $X_1, X_2, \ldots, X_n$ mutually independent,

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i}^{n} \mathrm{Var}(X_i)$$

(Not Covered) Proof of $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Theorem. If X, Y independent, $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Proof

Let $x_i, y_i, i = 1, 2, \ldots$ be the possible values of X, Y.

$$\mathbb{E}[X \cdot Y] = \sum_i \sum_j x_i \cdot y_j \cdot P(X = x_i \wedge Y = y_j)$$

independence

$$= \sum_i \sum_j x_i \cdot y_i \cdot P(X = x_i) \cdot P(Y = y_j)$$

$$= \sum_i x_i \cdot P(X = x_i) \cdot \left(\sum_j y_j \cdot P(Y = y_j) \right)$$

$$= \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Note: *NOT* true in general; see earlier example $\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$

(Not Covered) Proof of $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Theorem. If X, Y independent, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Proof

$$Var(X+Y)$$
$$= \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2$$
$$= \mathbb{E}[X^2 + 2XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2$$
$$= \mathbb{E}[X^2] + 2\,\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X]^2 + 2\,\mathbb{E}[X]\,\mathbb{E}[Y] + \mathbb{E}[Y]^2)$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2 + \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y]$$
$$= Var(X) + Var(Y) + 2\,\mathbb{E}[XY] - 2\,\mathbb{E}[X]\,\mathbb{E}[Y]$$
$$= Var(X) + Var(Y)$$

linearity

equal by independence

Recap Random Variables and Independence

Comma is shorthand for AND

Definition. Two random variables X, Y are **(mutually) independent** if for all x, y,

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$$

Intuition: Knowing X doesn't help you guess Y and vice versa

Definition. The random variables $X_1, \dots, X_n$ are **(mutually) independent** if for all $x_1, \dots, x_n$,

$$P(X_1 = x_1, \dots, X_n = x_n) = P(X_1 = x_1) \cdots P(X_n = x_n)$$

Note: No need to check for all subsets, but need to check <u>for all values</u>!

Example – Coin Tosses

We flip n *independent* coins, each one heads with probability p

- $X_i = \begin{cases} 1, & i^{\text{th}} \text{ outcome is heads} \\ 0, & i^{\text{th}} \text{ outcome is tails.} \end{cases}$
- $Z =$ number of heads

Fact. $Z = \sum_{i=1}^{n} X_i$

$P(X_i = 1) = p$

$P(X_i = 0) = 1 - p$

$\mathbb{E}(X_i) = p$

By LOE $\mathbb{E}[Z] = \sum_{i=1}^{n} \mathbb{E}(X_i) = np$

$P(Z = k) =$

$\text{Var}(X_i) =$

$\text{Var}(Z) =$

Example – Coin Tosses (2)

We flip n *independent* coins, each one heads with probability p

- $X_i = \begin{cases} 1, & i^{\text{th}} \text{ outcome is heads} \\ 0, & i^{\text{th}} \text{ outcome is tails.} \end{cases}$
- $Z =$ number of heads

Fact. $Z = \sum_{i=1}^{n} X_i$

$P(X_i = 1) = p$
$P(X_i = 0) = 1 - p$
$\mathbb{E}(X_i) = p$

$P(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$

By LOE $\mathbb{E}[Z] = \sum_{i=1}^{n} \mathbb{E}(X_i) = np$

Note: $X_1, \ldots, X_n$ are mutually independent!

$$\text{Var}(Z) = \sum_{i=1}^{n} \text{Var}(X_i) = n \cdot p(1-p)$$

Note $\text{Var}(X_i) = p(1-p)$

Agenda (if time)

- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Motivation for “Named” Random Variables

Random Variables that show up all over the place.

- Easily solve a problem by recognizing it’s a special case of one of these random variables.

Each RV introduced today will show:

- A general situation it models
- Its name and parameters
- Its PMF, Expectation, and Variance
- Example scenarios you can use it

Welcome to the Zoo! (Preview) 🦍🐘🦁🐅🦓🐪🦒

$X \sim \mathrm{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$

$$\mathbb{E}[X] = \frac{a + b}{2}$$

$$\mathrm{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \mathrm{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$\mathbb{E}[X] = p$$

$$\mathrm{Var}(X) = p(1 - p)$$

$X \sim \mathrm{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\mathbb{E}[X] = np$$

$$\mathrm{Var}(X) = np(1 - p)$$

$X \sim \mathrm{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$

$$\mathbb{E}[X] = \frac{1}{p}$$

$$\mathrm{Var}(X) = \frac{1 - p}{p^2}$$

$X \sim \mathrm{NegBin}(r, p)$

$$P(X = k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$

$$\mathbb{E}[X] = \frac{r}{p}$$

$$\mathrm{Var}(X) = \frac{r(1 - p)}{p^2}$$

$X \sim \mathrm{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$\mathbb{E}[X] = n\frac{K}{N}$$

$$\mathrm{Var}(X) = n\frac{K(N - K)(N - n)}{N^2(N - 1)}$$

Zoo of Discrete RVs, Part I

- Uniform Random Variables
- Bernoulli Random Variables
- Binomial Random Variables
- Geometric Random Variables

Discrete Uniform Random Variables

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is uniform.

Notation:

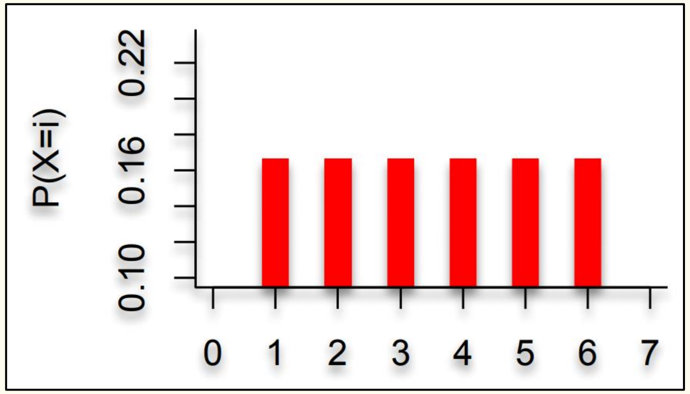
PMF:

Expectation:

Variance:

Example: value shown on one roll of a fair die is $\text{Unif}(1,6)$:

- $P(X = i) = 1/6$
- $\mathbb{E}[X] = 7/2$
- $\text{Var}(X) = 35/12$



Discrete Uniform Random Variables (2)

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is uniform.

Notation: $X \sim \mathrm{Unif}(a, b)$

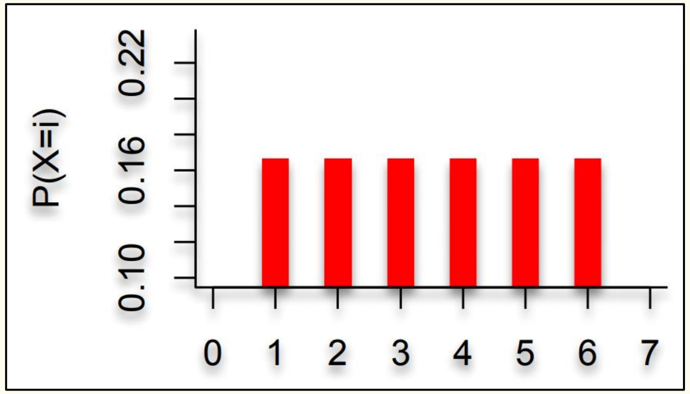
PMF: $\mathrm{P}(X = i) = \frac{1}{b - a + 1}$

Expectation: $\mathbb{E}[X] = \frac{a+b}{2}$

Variance: $\mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$

Example: value shown on one roll of a fair die is $\mathrm{Unif}(1,6)$:

- $P(X = i) = 1/6$
- $\mathbb{E}[X] = 7/2$
- $\mathrm{Var}(X) = 35/12$



Zoo (2)

- Zoo of Discrete RVs
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Bernoulli Random Variables

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \text{Ber}(p)$

PMF: $P(X = 1) = p,\ P(X = 0) = 1 - p$

Expectation:

Variance:

Poll:
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	Mean	Variance
A.	p	p
B.	p	$1 - p$
C.	p	$p(1 - p)$
D.	p	p^2

Bernoulli Random Variables (2)

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \mathrm{Ber}(p)$

PMF: $P(X = 1) = p,\ P(X = 0) = 1 - p$

Expectation: $\mathbb{E}[X] = p$ Note: $\mathbb{E}[X^2] = p$

Variance: $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = p(1 - p)$

Examples:

- Coin flip
- Randomly guessing on a MC test question
- A server in a cluster fails
- Whether or not a share of a particular stock pays off or not
- Any indicator r.v.

Zoo (3)

- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Binomial Random Variables

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \mathrm{Ber}(p)$.

Counts number of successes in n independent trials, each with probability p of success.

X is a Binomial random variable

Examples:

- # of heads in n indep coin flips
- # of 1s in a randomly generated n bit string
- # of servers that fail in a cluster of n computers
- # of bit errors in file written to disk
- # of elements in a bucket of a large hash table
- # of n different stocks that "pay off"

Poll:
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$\boldsymbol{P(X = k) =}$

A. $p^k(1-p)^{n-k}$
B. np
C. $\binom{n}{k}p^k(1-p)^{n-k}$
D. $\binom{n}{n-k}p^k(1-p)^{n-k}$

Binomial Random Variables (2)

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \mathrm{Ber}(p)$.
Counts number of successes in n independent trials, each with probability p of success.

X is a Binomial random variable

Poll:
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	Mean	Variance
A.	p	p
B.	np	$np(1-p)$
C.	np	np^2
D.	np	n^2p

Notation: $X \sim \mathrm{Bin}(n, p)$

PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation:

Variance:

Binomial Random Variables (3)

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \mathrm{Ber}(p)$.

Counts number of successes in n independent trials, each with probability p of success.

X is a Binomial random variable

Notation: $X \sim \mathrm{Bin}(n, p)$

PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation: $\mathbb{E}[X] = np$

Variance: $\mathrm{Var}(X) = np(1-p)$

Mean, Variance of the Binomial

"i.i.d." is a commonly used phrase.
It means "independent & identically distributed"

If $Y_1, Y_2, \dots, Y_n \sim \text{Ber}(p)$ and independent (i.i.d.), then
$X = \sum_{i=1}^{n} Y_i, \quad X \sim \text{Bin}(n, p)$

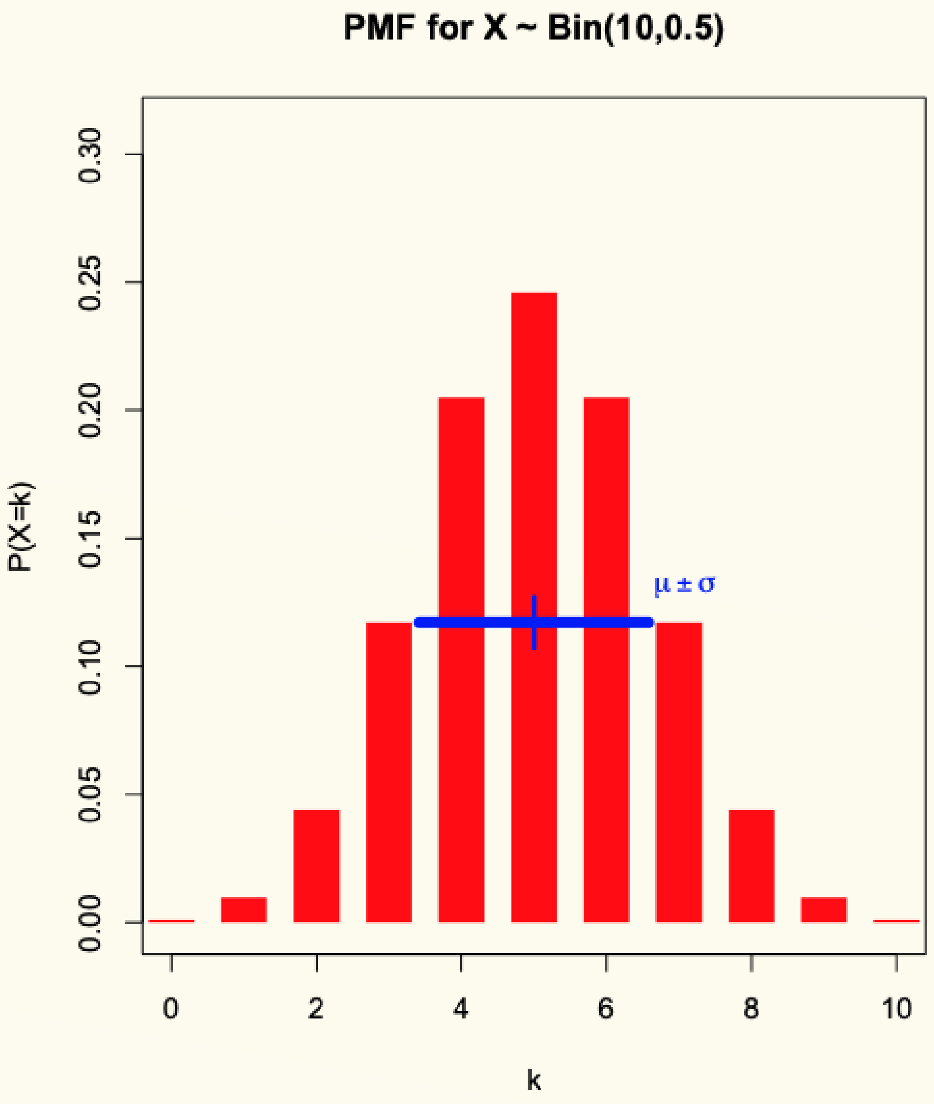
Claim $\mathbb{E}[X] = np$

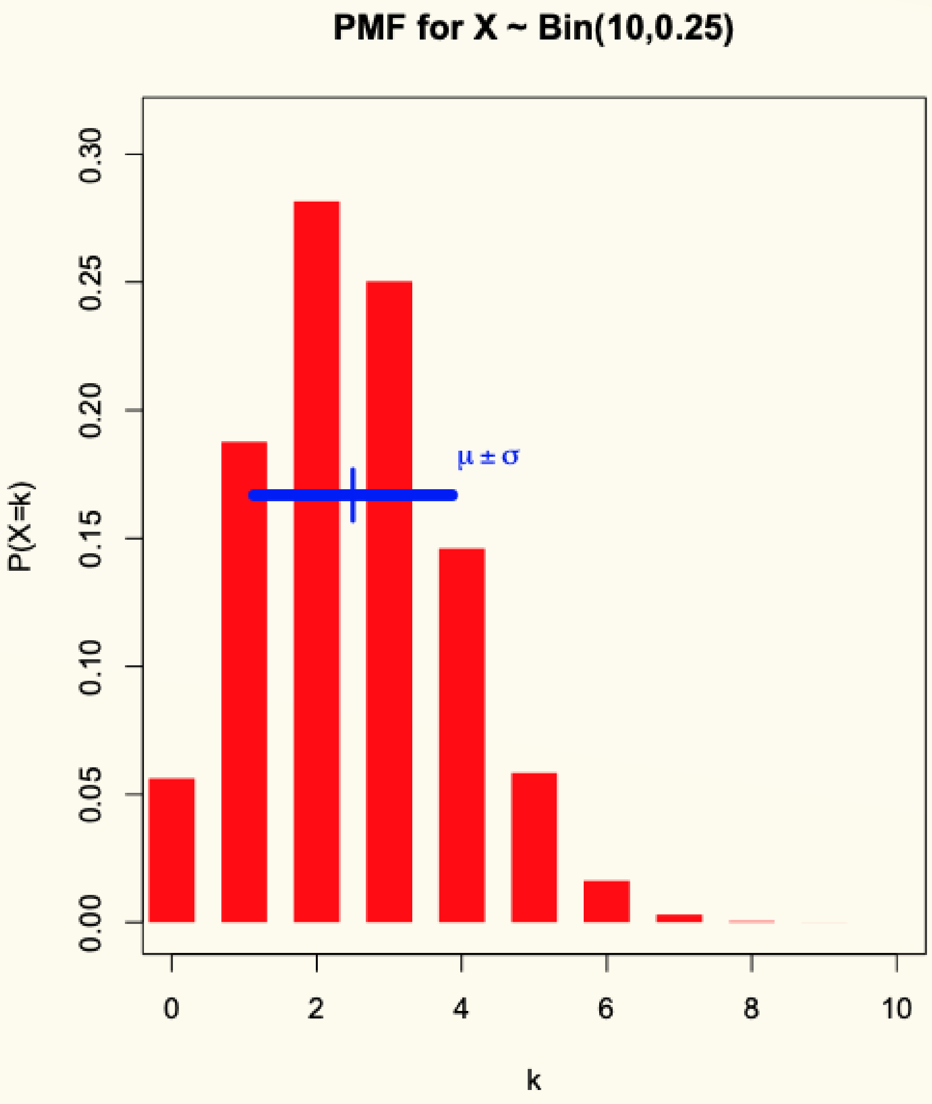
$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = n\mathbb{E}[Y_1] = np$$

Claim $\text{Var}(X) = np(1-p)$, by independence of r.v.'s

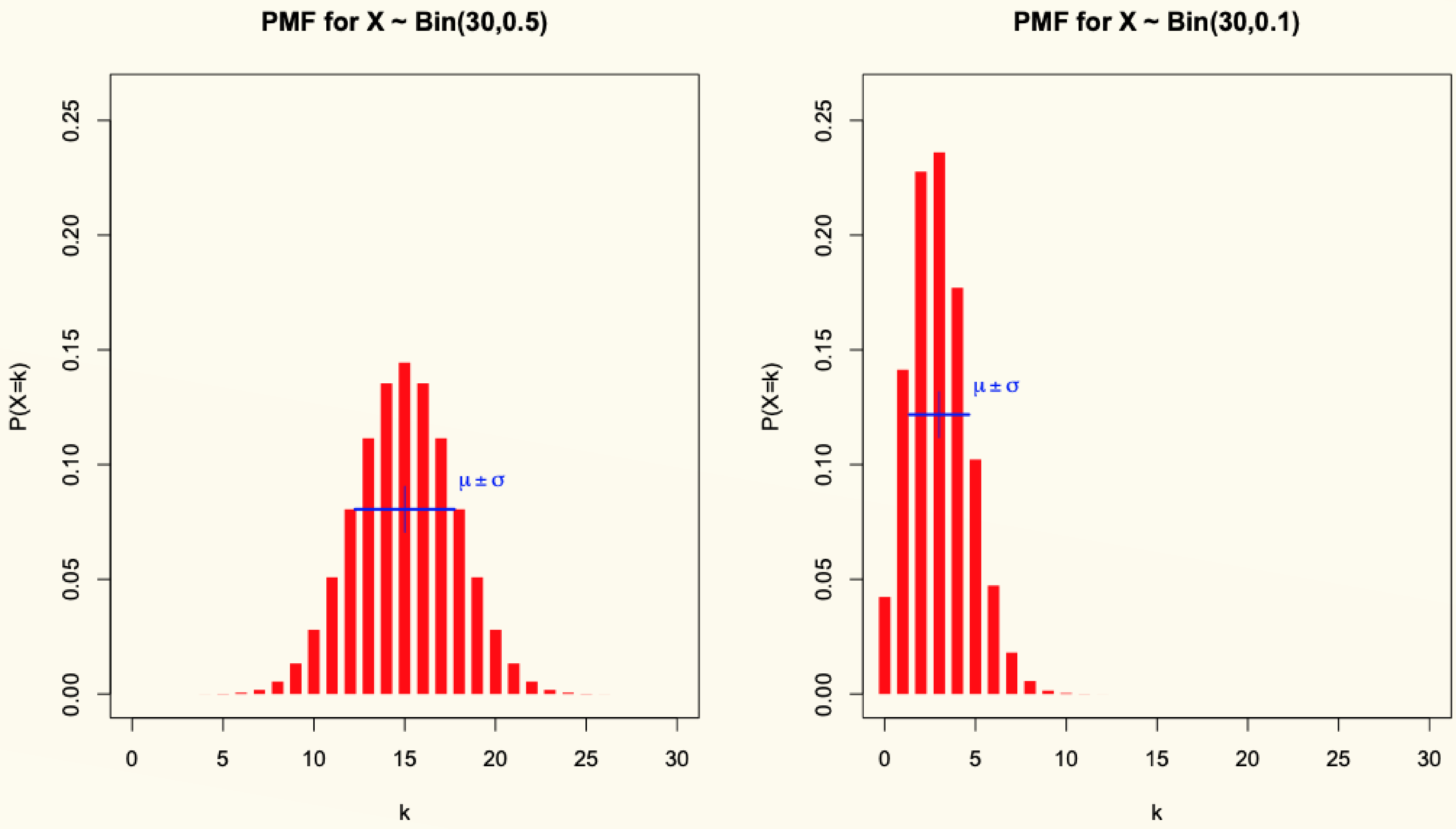
$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \text{Var}(Y_i) = n\text{Var}(Y_1) = np(1-p)$$

Binomial PMFs





Binomial PMFs (2)



Zoo (4)

- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables

Geometric Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim \mathrm{Ber}(p)$ up until and including the first success.

X is called a Geometric random variable with parameter p.

Notation: $X \sim \mathrm{Geo}(p)$

PMF:

Expectation:

Variance:

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

Geometric Random Variables (2)

A discrete random variable X that models the number of independent trials $Y_i \sim \mathrm{Ber}(p)$ up until and including the first success.

X is called a Geometric random variable with parameter p.

Notation: $X \sim \mathrm{Geo}(p)$

PMF: $P(X = k) = (1 - p)^{k-1} p$

Expectation: $\mathbb{E}[X] = \frac{1}{p}$

Variance: $\mathrm{Var}(X) = \frac{1-p}{p^2}$

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it
- # hash trials until a miner successfully mines a Bitcoin

Zoo (5)

- Zoo of Discrete RVs, Part I
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables
  - More examples

Example 1

Sending a binary message of length 1024 bits over a network with probability 0.999 of correctly sending each bit in the message without corruption (independent of other bits).

Let X be the number of corrupted bits.

What kind of random variable is this and what is $\mathbb{E}[X]$?

Poll:
https://pollev.com/ annakarlin185

A 1022.99
B 1.024
C 1.02298
D. 1

Example: Music Lessons

Your music teacher requires you to play a 1000 note song without mistake. You have been practicing, so you have a probability of 0.999 of getting each note correct (independent of the others). If you mess up a single note in the song, you must start over and play from the beginning. Let X be the number of times you have to play the song from the start. What kind of random variable is this and what is $\mathbb{E}[X]$?