# Linearity of Expectation, LOTUS and Variance

CSE 312 Spring 26

Lecture 10

## Review Random Variables and Associated Events

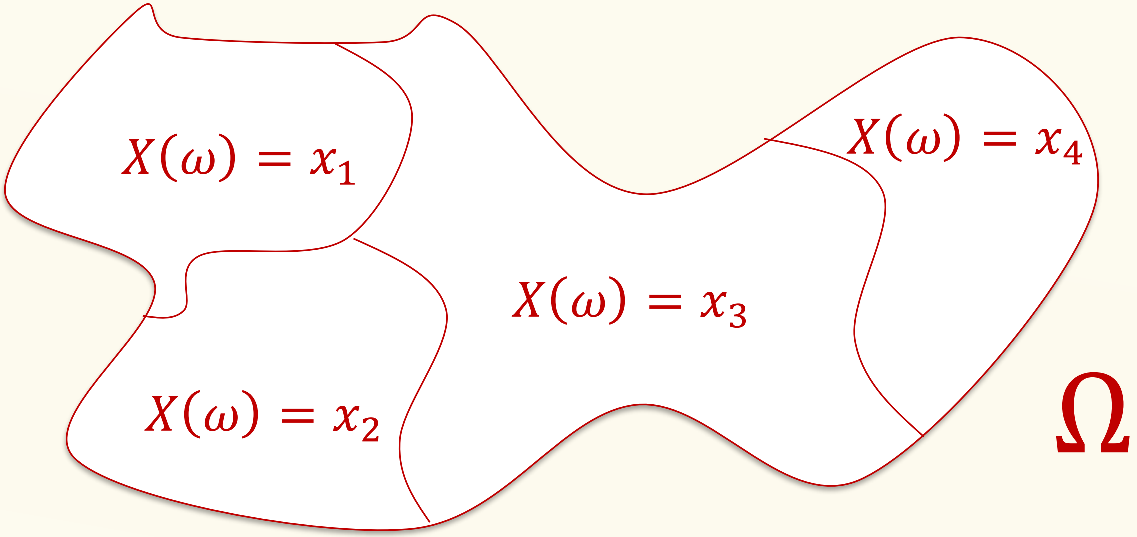
**Definition.** A **random variable (RV)** for a probability space $(\Omega, P)$ is a function $X : \Omega \to \mathbb{R}$.

The set of values that $X$ can take on is its *range/support*: $\boldsymbol{\Omega}_{\mathrm{X}}$

$\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$

Random variables **partition** the sample space.

$\Sigma_{x \in \boldsymbol{\Omega}_{\mathrm{X}}} P(X = x) = 1$



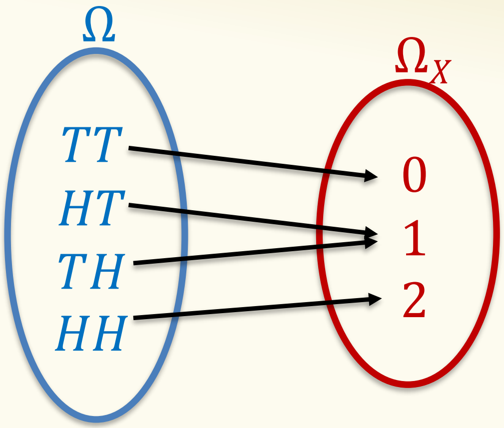
# Example

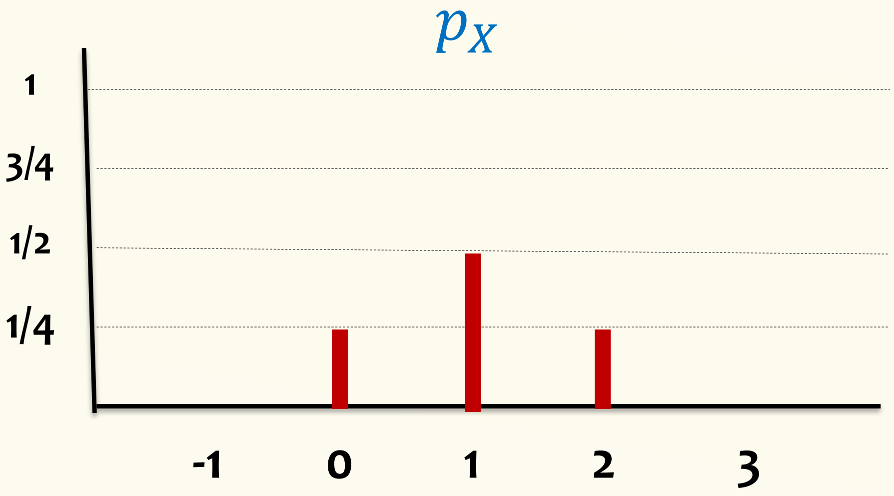
Two fair coin flips

$$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$$

$$X = \text{number of heads}$$

$$\Omega_X = \{0,1,2\}$$




$$p_X(0) = \Pr(X = 0) = \Pr(\mathrm{TT})$$

$$p_X(1) = \Pr(X = 1) = \Pr(\{\mathrm{TH}, \mathrm{HT}\})$$

$$p_X(2) = \Pr(X = 2) = \Pr(\mathrm{HH})$$

# Example – Two fair independent coin flips

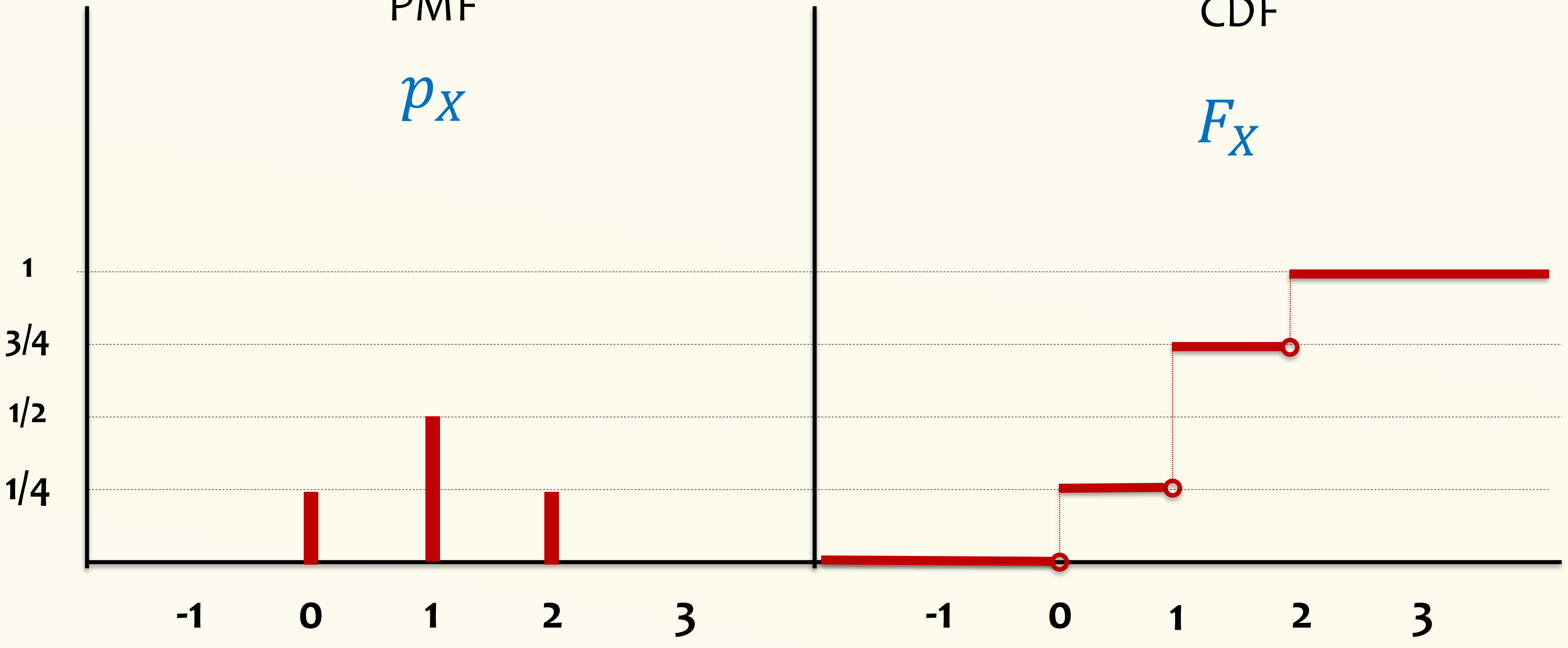
$X$ = number of heads

Probability Mass Function
PMF

$p_X$

Cumulative Distribution Function
CDF

$F_X$



## Review PMF and CDF

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \le x < 1 \\ \frac{3}{4} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

**Definitions:**

For a RV $X: \Omega \to \mathbb{R}$, the probability mass function (pmf) of $X$ specifies, for any real number $x$, the probability that $X = x$

$$p_X(x) = P(X = x) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\boxed{\sum_{x \in \Omega_X} p_X(x) = 1}$$

For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function (cdf) of $X$ specifies, for any real number $x$, the probability that $X \le x$

$$F_X(x) = P(X \le x)$$

$$\boxed{\Omega_X = \{0, 1, 2, \ldots, 10\}}$$

$$p_X(x)$$

$$F_X(4) = \sum_{x \le 4} p_X(x)$$

$$F_X(x) \quad \forall x \in \mathbb{R}$$

$$p_X(5) = F_X(5) - F_X(4)$$

## Review Expected Value of a Random Variable

**Definition.** Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $X$ is

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x) \qquad = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

$$X = \begin{cases} 5 & \text{w/ prob } \frac{1}{5} \\ 10 & \text{w/ prob } \frac{4}{5} \end{cases}$$

$$E(X) = 5 \cdot \frac{1}{5} + 10 \cdot \frac{4}{5} = 9$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

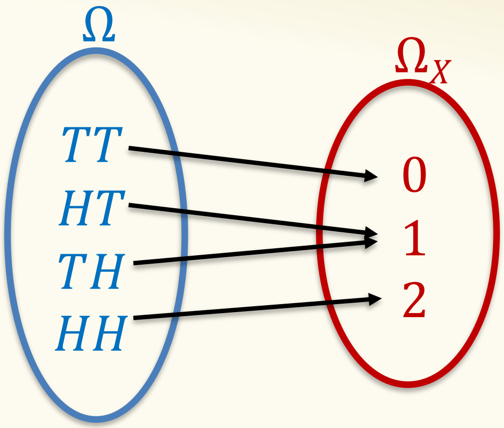
# Example calculation

Two fair coin flips

$$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$$

$$X = \text{number of heads}$$

$$\Omega_X = \{0,1,2\}$$

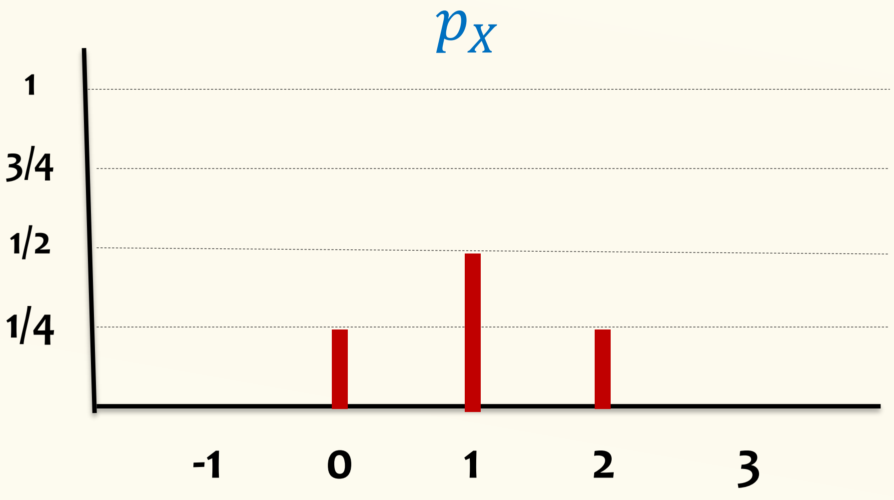



$$\mathbb{E}[\boldsymbol{X}] = \sum_{x \in \Omega_X} \boldsymbol{x} \cdot \boldsymbol{P}(\boldsymbol{X} = \boldsymbol{x})$$

$$p_X(0) = \Pr(X = 0) = \Pr(\mathrm{TT})$$

$$p_X(1) = \Pr(X = 1) = \Pr(\{\mathrm{TH}, \mathrm{HT}\})$$

$$p_X(2) = \Pr(X = 2) = \Pr(\mathrm{HH})$$




$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

# Expected Value of a Random Variable

**Definition.** Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $X$ is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x) \quad = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

# Expectation

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

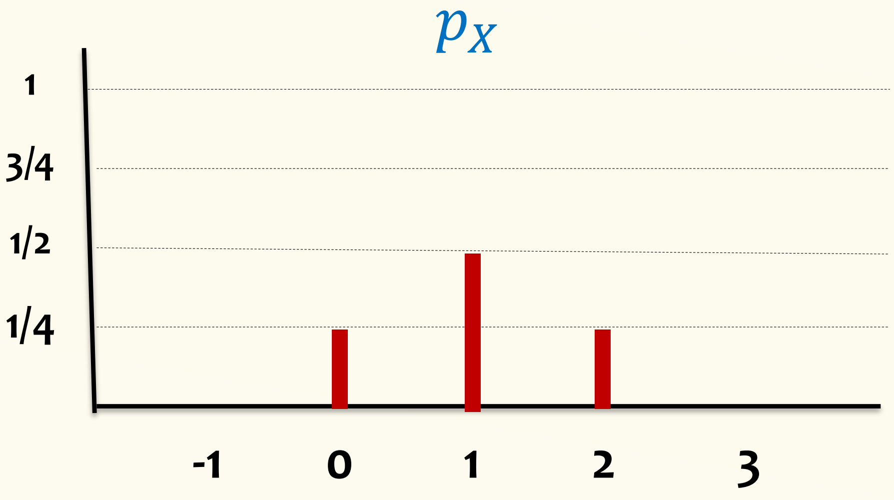
$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

**Example.** Two fair coin flips

$$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$$

$X =$ number of heads

What is $\mathbb{E}[X]$?




$$\mathbb{E}[X] = X(TT)\,P(TT) + X(HT)\,P(HT) + X(TH)\,P(TH) + X(HH)\,P(HH)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = 1$$

$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

# Recap Linearity of Expectation

**Theorem.** For any two random variables $X$ and $Y$

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Or, more generally: For any random variables $X_1, \ldots, X_n$,

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

**Theorem.** For any random variables $X$, and constants $a$ and $b$

$$\mathbb{E}[aX+b] = a \cdot \mathbb{E}[X] + b.$$

$X$

$Y = 3X - 5$

$$E(Y) = E(3X-5)$$
$$= 3E(X) - 5$$

# Recap Linearity of Expectation (more)

**Theorem.** For any two random variables $X$ and $Y$

$$\mathbb{E}[X+Y]=\mathbb{E}[X]+\mathbb{E}[Y].$$

Or, more generally: For any random variables $X_1, \dots, X_n$,

$$\mathbb{E}[X_1+\cdots+X_n]=\mathbb{E}[X_1]+\cdots+\mathbb{E}[X_n].$$

**Theorem.** For any random variables $X$, and constants $a$ and $b$

$$\mathbb{E}[aX+b]=a\cdot\mathbb{E}[X]+b.$$

**Theorem.** For any random variables $X_1, \dots, X_n$, and real numbers $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}[a_1X_1+\cdots+a_nX_n]=a_1\mathbb{E}[X_1]+\cdots+a_n\mathbb{E}[X_n].$$

# Agenda

- Recap
- Linearity of expectation
- LOTUS
- Variance

# Pairs with the same birthday

- In a class of $m$ students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?
- Each person's birthday is equally likely to be any of the 365 possibilities and different people's bdays are independent.

$m=6$

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Aug 1 | Oct 21 | Oct 21 | Oct 21 | Nov 1 | Nov 1 |

2,3   3,4   5,6   2,4

$X$ : # pairs of people w/ same bday

$\Omega_X = \{0, 1, \ldots, \binom{m}{2}\}$

$\binom{m}{\ }$

$$E(X) = \sum_{k=0}^{(\infty)} k \, P(X=k)$$

# Pairs with the same birthday - LOE

- In a class of $m$ students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?
- Each person's birthday is equally likely to be any of the 365 possibilities and different people's bdays are independent.

Decompose: Find the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Conquer: Compute the expectation of each $X_i$ and sum!

$X$ : # pairs of people w/ same bday

$\Omega_X = \{0, 1, \dots, \binom{m}{2}\}$

$$E(X) = \sum_{k=0}^{\binom{m}{2}} k\, P(X=k)$$

$X_{ij} = \begin{cases} 1 & \text{person } i \text{ \& person } j \text{ have same bday} \\ 0 & \text{o.w.} \end{cases}$

$$E[X] = \sum_{1 \le i < j \le m} E[X_{ij}] = \binom{m}{2} \frac{1}{365}$$

$$E(X_{ij}) = P(\text{person } i \text{ \& person } j \text{ have same bday})$$

$$= \sum_{b=1}^{365} \underbrace{P(\text{person } i \text{ \& person } j \text{ born on day } b)}_{P(\text{person } i \text{ has bday } b)\, P(\text{person } j \text{ has bday } b)}$$

$\frac{1}{365}$ $\frac{1}{365}$

$$= 365 \cdot \frac{1}{365} \cdot \frac{1}{365} = \frac{1}{365}$$

# Pairs with the same birthday – LOE (2)

- In a class of $m$ students, let $X$ be the number of pairs of people with the same birthday (assuming 365 equally likely birthdays)?
- What is $\mathbb{E}[X]$?

Decompose: Indicator events involve **pairs** of students $(i, j)$ for $i \neq j$

$X_{ij} = 1$ iff students $i$ and $j$ have the same birthday

LOE: $\binom{m}{2}$ indicator variables $X_{ij}$

Conquer: $\mathbb{E}[X_{ij}] = \frac{1}{365}$ so total expectation is $\frac{\binom{m}{2}}{365}$ pairs

# Agenda (2)

- Recap
- Linearity of expectation
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- Variance

# Constructing new random variables from old ones.

1. $X \rightarrow Y = 2X - 1$
2. $X_1, X_2, \ldots, X_n \rightarrow Z = X_1 + \cdots + X_n$
3. $X_1, X_2 \rightarrow W = 5X_1 - 3X_2 + 2$   $E(W) = 5E(X_1) - 3E(X_2) + 2$
4. $X \rightarrow U = X^3 \bmod 4$

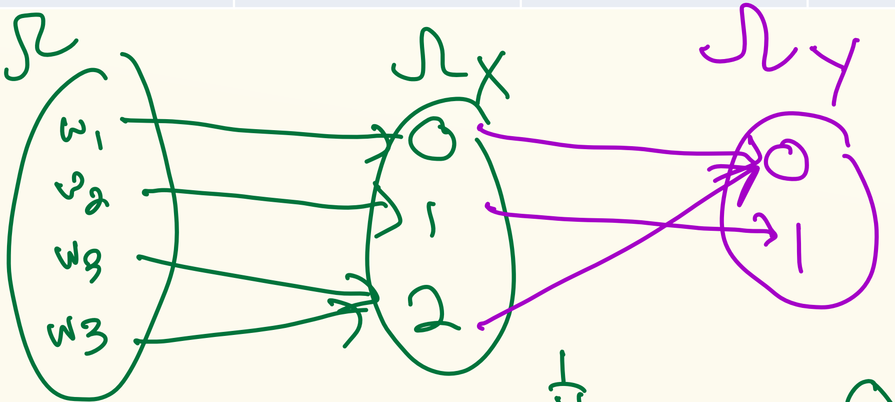
$$\cancel{E(U) = E(X)^3 \bmod 4}$$

How do we compute the expectation of the new r.v.s, given the expectations of the old r.v.s?

# Example of constructing new r.v.

$g(x) = x^2 \bmod 4.$

| $\Pr(\omega)$ | $\omega$ | $X$ | $Y = X^2 \bmod 4$ |
|---|---|---|---|
| 1/4 | $\omega_1$ | 0 | 0 |
| 1/6 | $\omega_2$ | 1 | 1 |
| 1/4 | $\omega_3$ | 2 | 0 |
| 1/3 | $\omega_4$ | 2 | 0 |




$$P_Y(y) = \begin{cases} \frac{5}{6} & 0 \\ \frac{1}{6} & 1 \end{cases}$$

$$P_X(x) = \begin{cases} \frac{1}{4} & 0 \\ \frac{1}{6}, & 1 \\ \frac{1}{4}+\frac{1}{3}=\frac{7}{12} & 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[Y] = \sum_{y \in \Omega_Y} y \cdot P(Y = y)$$

$$\mathbb{E}[\overset{x^2 \bmod 4}{g(X)}] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$$

$$E(Y) = g(0)\cdot\frac{1}{4} + g(1)\cdot\frac{1}{6} + g(2)\cdot\frac{7}{12}$$

$$= (0^2 \bmod 4)\frac{1}{4} + (1^2 \bmod 4)\frac{1}{6} + (2^2 \bmod 4)\frac{7}{12}$$

# Expected Value of $g(X)$

**Definition.** Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $Y = g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{\omega \in \Omega} g\big(X(\omega)\big) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x) = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Also known as **LOTUS:** "Law of the unconscious statistician

# Linearity is special!

$\mathbb{E}[\boldsymbol{X}] = \sum_{x \in \Omega_X} \boldsymbol{x} \cdot \boldsymbol{P}(\boldsymbol{X} = \boldsymbol{x})$

$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$

In general $\mathbb{E}[g(X)] \neq g\big(\mathbb{E}(X)\big)$

E.g., $X = \begin{cases} +1 \text{ with prob } 1/2 \\ -1 \text{ with prob } 1/2 \end{cases}$

$g(x) = x^2$

$\mathbb{E}[X] = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$

$\mathbb{E}[X^2] = 1^2 \cdot \frac{1}{2} + (-1)^2 \frac{1}{2} = 1$

$\mathbb{E}[X]^2 = 0^2 = 0$

$1 \quad 0$

$\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$

# Agenda (3)

- Recap
- Linearity of expectation
- LOTUS
- Variance

## A game

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

***Game 1:*** *In every round, you win \$2 with probability 1/3, lose \$1 with probability 2/3.*

$W_1$ = payoff in a round of Game 1

$P(W_1 = 2) = \frac{1}{3}$, $P(W_1 = -1) = \frac{2}{3}$

$E(W_1) = 2 \cdot \frac{1}{3} + (-1) \cdot \frac{2}{3} = 0$

# Which game would you rather play?

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

***Game 1:*** *In every round, you win $2 with probability 1/3, lose $1 with probability 2/3.*

$W_1$ = payoff in a round of Game 1

$P(W_1 = 2) = \frac{1}{3}, P(W_1 = -1) = \frac{2}{3}$

$\mathbb{E}[W_1] = 0$

***Game 2:*** *In every round, you win $10 with probability 1/3, lose $5 with probability 2/3.*

$E(W_2) = 10 \cdot \frac{1}{3} + (-5) \cdot \frac{2}{3} = 0$

$W_2$ = payoff in a round of Game 2

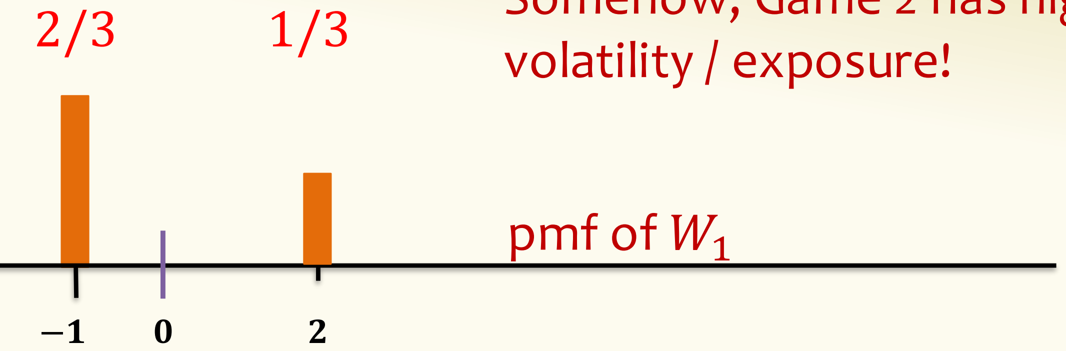
$P(W_2 = 10) = \frac{1}{3}, P(W_2 = -5) = \frac{2}{3}$

$\mathbb{E}[W_2] = 0$

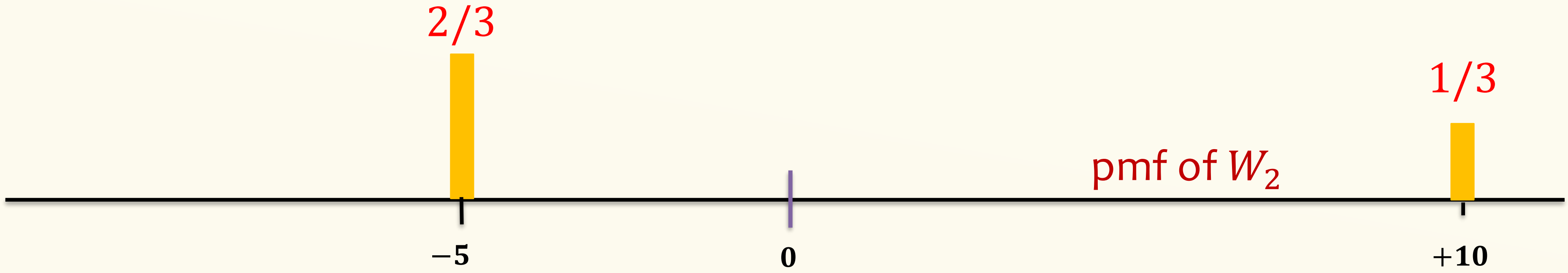
# Two Games

Somehow, Game 2 has higher volatility / exposure!

$P(W_1 = 2) = \frac{1}{3}$ , $P(W_1 = -1) = \frac{2}{3}$




$P(W_2 = 10) = \frac{1}{3}$ , $P(W_2 = -5) = \frac{2}{3}$



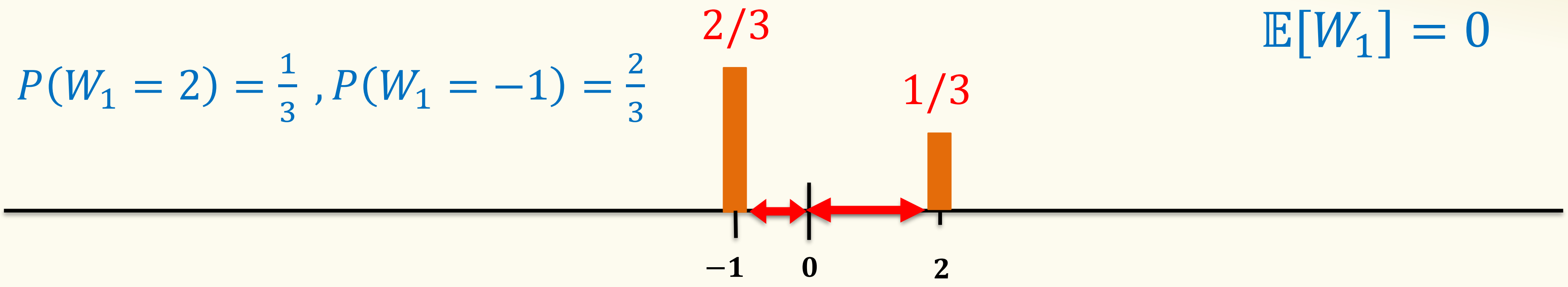

Same expectation, but clearly a very different distribution.

We want to capture the difference – New concept: **Variance**

# Variance (Intuition, First Try)

$P(W_1 = 2) = \frac{1}{3}, P(W_1 = -1) = \frac{2}{3}$

$\mathbb{E}[W_1] = 0$




New quantity (random variable): How far from the expectation?

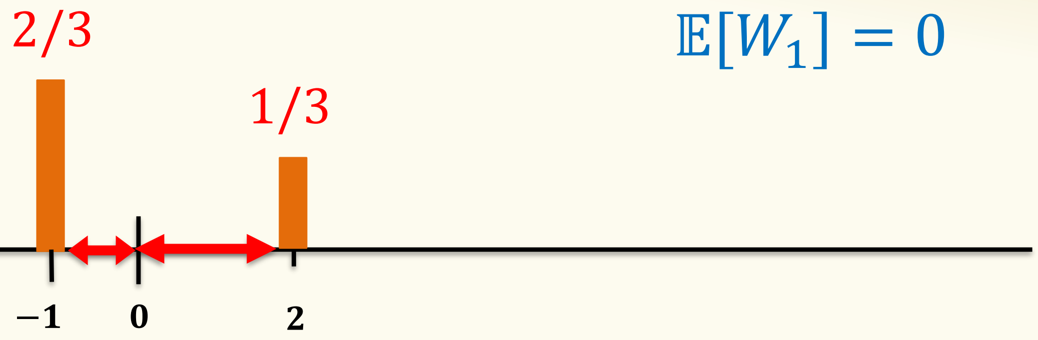
$W_1 - \mathbb{E}[W_1]$

$$E[W_1 - E(W_1)]$$
$$= E(W_1) - E(E(W_1))$$
$$= E(W_1) - E(W_1) = 0$$

# Variance (Intuition, First Try with calculation)

$P(W_1 = 2) = \frac{1}{3}, P(W_1 = -1) = \frac{2}{3}$

$\mathbb{E}[W_1] = 0$




New quantity (random variable): How far from the expectation?

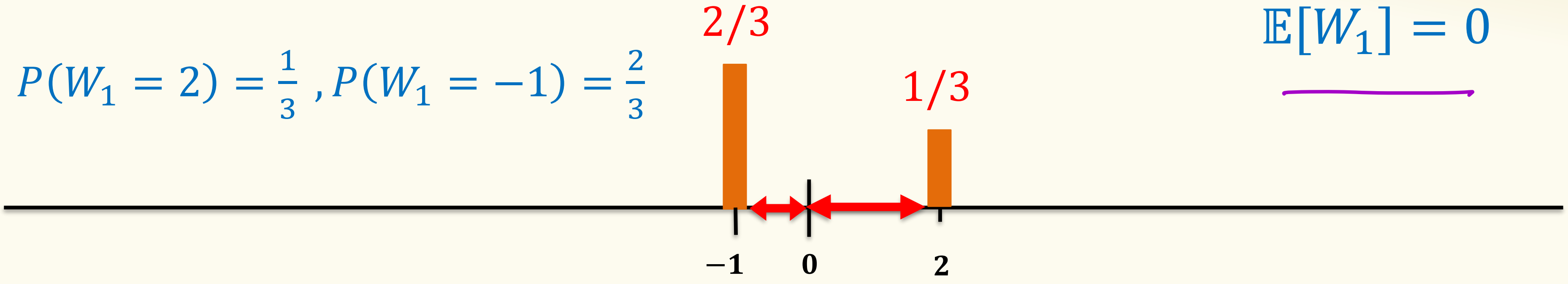
$W_1 - \mathbb{E}[W_1]$

$$\begin{aligned}
&\mathbb{E}[W_1 - \mathbb{E}[W_1]] \\
&= \mathbb{E}[W_1] - \mathbb{E}\big[\mathbb{E}[W_1]\big] \\
&= \mathbb{E}[W_1] - \mathbb{E}[W_1] \\
&= 0
\end{aligned}$$

# Variance (Intuition, Better Try)

$P(W_1 = 2) = \frac{1}{3}, P(W_1 = -1) = \frac{2}{3}$

$\mathbb{E}[W_1] = 0$




New quantity (random variable): How far from the expectation?

$(W_1 - \mathbb{E}[W_1])^2$

$g(x) = [x - E[W_1]]^2$

$E(g(W_1))$

$\mathbb{E}[(W_1 - \mathbb{E}[W_1])^2]$

$= (2 - E(W_1))^2 \cdot \frac{1}{3} + (-1 - E(W_1))^2 \cdot \frac{2}{3}$

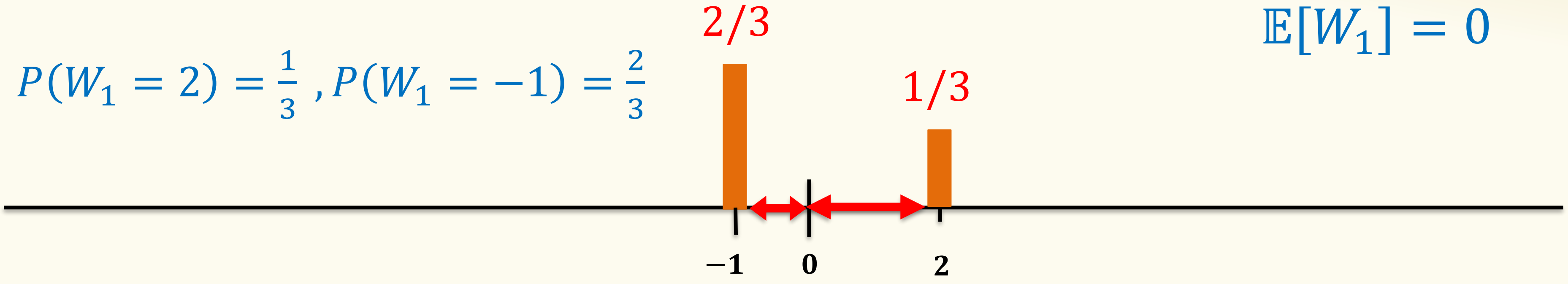
$= 2^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{2}{3} = 2$

$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$

# Variance (Intuition, Better Try with calculation)

$P(W_1 = 2) = \frac{1}{3}, P(W_1 = -1) = \frac{2}{3}$

$\mathbb{E}[W_1] = 0$




New quantity (random variable): How far from the expectation?

$\left(W_1 - \mathbb{E}[W_1]\right)^2$

$$\mathbb{E}[(W_1 - \mathbb{E}[W_1])^2]$$

$$= \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 4$$

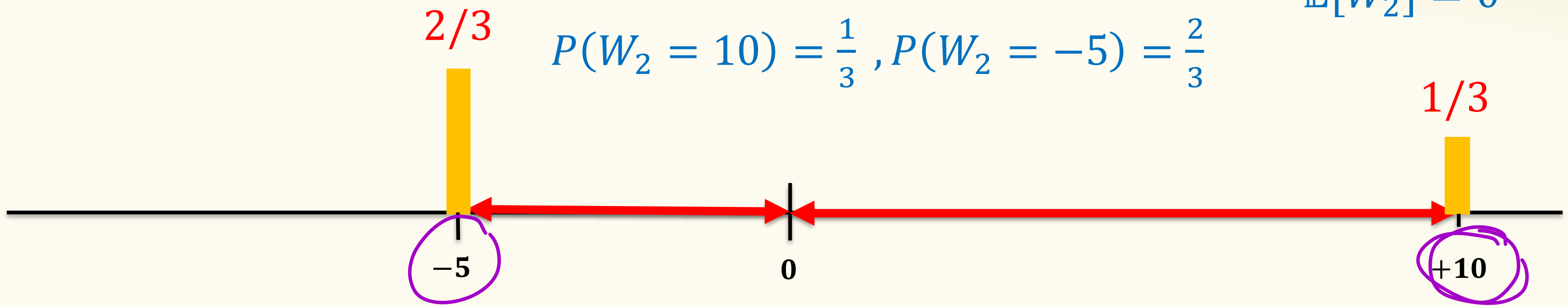
$$= 2$$

$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$

# Variance (Intuition, Better Try – $W_2$)

$$\mathbb{E}[W_2] = 0$$

$$P(W_2 = 10) = \frac{1}{3},\ P(W_2 = -5) = \frac{2}{3}$$




A better quantity (random variable): How far from the expectation?

$$(10 - \mathbb{E}(W_2))^2 \cdot \frac{1}{3}$$
$$+ (-5 - \mathbb{E}(W_2))^2 \cdot \frac{2}{3}$$

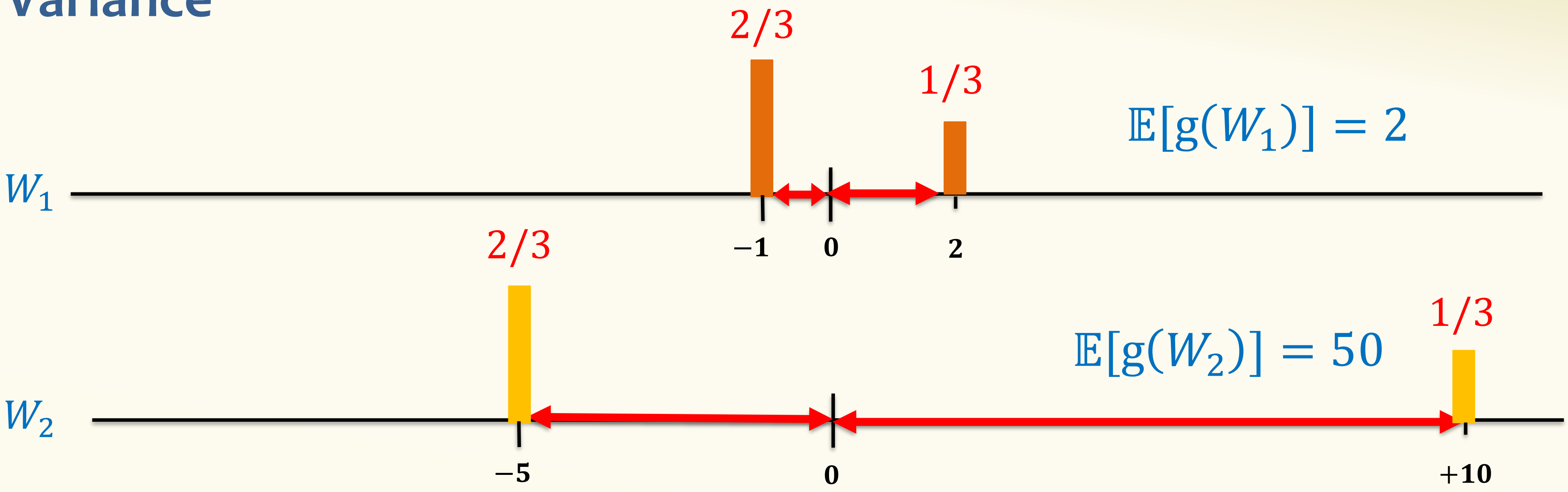
$$\mathbb{E}[(W_2 - \mathbb{E}[W_2])^2]$$

$$= \frac{2}{3} \cdot 25 + \frac{1}{3} \cdot 100$$

$$= 50$$

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$$

# Variance




We say that $W_2$ has **"higher variance"** than $W_1$.

$E(|w - E(w)|)$

$$g(W) = (W - \mathbb{E}[W])^2$$

# Variance - summary

**Definition.** The **variance** of a (discrete) RV $X$ is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \Omega_X} p_X(x) \cdot (x - \mathbb{E}[X])^2$$

Recall $\mathbb{E}[X]$ is a **constant**, not a random variable itself.

**Standard deviation:** $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

**Intuition:** Variance (or standard deviation) is a quantity that measures, in expectation, how "far" the random variable is from its expectation.

# Variance – Example 1

$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$

$X$ fair die

- $P(X = 1) = \cdots = P(X = 6) = 1/6$
- $\mathbb{E}[X] = 3.5$

$g(x) = (x - \mathbb{E}(X))^2$

$$\mathrm{Var}(X) = \sum_{x \in \Omega_X} P(X = x) \cdot (x - \mathbb{E}[X])^2$$

$$= \frac{1}{6}(1 - 3.5)^2 + \frac{1}{6}(2 - 3.5)^2 + \frac{1}{6}(3 - 3.5)^2$$

$$+ \ldots \frac{1}{6}(6 - 3.5)^2$$

# Variance – Example 1 (with calculation)

$X$ fair die

- $P(X = 1) = \cdots = P(X = 6) = 1/6$
- $\mathbb{E}[X] = 3.5$

$$\mathrm{Var(X)} = \sum_x P(X = x) \cdot (x - \mathbb{E}[X])^2$$

$$= \frac{1}{6}[(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2]$$

$$= \frac{2}{6}[2.5^2 + 1.5^2 + 0.5^2] = \frac{2}{6}\left[\frac{25}{4} + \frac{9}{4} + \frac{1}{4}\right] = \frac{35}{12} \approx 2.91677 \ldots$$

# Variance in Pictures

Captures how much "spread' there is in a pmf

All pmfs have same expectation = 0