CSE 312 – Section 8

Spring 2026

Review of Main Concepts

- **Multivariate: Discrete to Continuous:**

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x, s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, dsdt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dxdy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dxdy$
Independence must have	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

- **Law of Total Probability (r.v. version):** If X is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A | X = x) p_X(x) \qquad \text{discrete } X$$

- **Continuous Law of Total Probability:**

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A | X = x) f_X(x) dx$$

There will be problems covering the following concepts (some of which have not yet been discussed in lecture) on the Section 9 worksheet:

- **Conditional expectation:** The expected value of random variable X given that event A has occurred, written $\mathbb{E}[X|A]$, is defined as

$$\mathbb{E}[X|A] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X = x | A).$$

- **Discrete Law of Total Expectation (event version):** Let $A_1 \dots, A_n$ be a partition of the sample space. Then

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X|A_i]\, \mathbb{P}(A_i).$$

- **Discrete Law of Total Expectation (r.v. version):** Let X and Y be two random variables. Then
$$\mathbb{E}[X] = \sum_{y\in\Omega_Y} \mathbb{E}[X|Y=y]\cdot \mathbb{P}(Y=y).$$
- **Continuous Law of Total Expectation:**
$$\mathbb{E}[X] = \int_{y\in\Omega_Y} \mathbb{E}[X|Y=y]\, f_Y(y)dy$$
- **Expected value of X conditioned on r.v. Y:** Suppose that Y is a random variable that takes values $y_1,\dots,y_k$. Then $\mathbb{E}[X|Y]$ is the following random variable
$$\mathbb{E}[X|Y] = \begin{cases} \mathbb{E}[X|Y=y_1] & \text{with probability } \mathbb{P}(Y=y_1) \\ \mathbb{E}[X|Y=y_2] & \text{with probability } \mathbb{P}(Y=y_2) \\ \dots & \\ \mathbb{E}[X|Y=y_k] & \text{with probability } \mathbb{P}(Y=y_k) \end{cases}$$
- **Law of total expectation (rewritten):** Given the above definition, we can write
$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \sum_{i=1}^{k} \mathbb{E}[X|Y=y_i]\cdot \mathbb{P}(Y=y_i).$$
- **Covariance:** We may not get to this in class, but there is a problem on the pset about it. To find out more, check out section 5.4 in the Tsun book. And now the definition: For any two random variables X, Y the *covariance* is defined as
$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])].$$
It can also be shown that
$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$
- **Conditional distributions:** We are not explicitly covering this topic in class, but it is **highly** recommended that you study it. Much of the above can be more appropriately rewritten in terms of conditional distributions. See Tsun, Section 5.3.

	Discrete	**Continuous**
Conditional PMF/PDF	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X\|Y=y] = \sum_x x p_{X\|Y}(x\|y)$	$\mathbb{E}[X\|Y=y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y)dx$

Plan for Section

- Content Review (Problem 1)
- Joint PMF's (Problem 2) - do it fast
- Continuous joint density - Problem 6

- 3 points on a line - Problem 7
- Min and max of i.i.d. random variables - Problem 8 if time permits

We recommend that students look at the final problem for examples of how to set up the ranges of integration. Might be helpful on the homework.

1 Content Review

a) Select one: Given two discrete random variables X and Y, the joint CDF is

○ $F_{X,Y}(x,y) = \sum_{t<x} p_{X,Y}(t,y)$
○ $F_{X,Y}(x,y) = \sum_{s<y} p_{X,Y}(x,s)$
○ $F_{X,Y}(x,y) = \sum_{t\leq x} \sum_{s\leq y} p_{X,Y}(t,s)$
○ $F_{X,Y}(x,y) = p_{X,Y}(x,y)$

b) **Marginal PDF.** Let X and Y be continuous random variables with joint PDF $f_{X,Y}(x,y)$. Which of the following correctly expresses the marginal PDF $f_X(x)$?

○ $\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$
○ $\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$
○ $\frac{f_{X,Y}(x,y)}{f_Y(y)}$
○ $\int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\,ds\,dt$

c) **Independence and Support.** True or False: If the joint support $\Omega_{X,Y}$ of the random variables (X,Y) is a circle defined by $x^2+y^2 \leq 1$, and $\Omega_X = \Omega_Y = [0,1]$ then X and Y are independent.

○ True
○ False

d) **Continuous Law of Total Probability.** Let A be an event and X be a continuous random variable with PDF $f_X(x)$. Which of the following is the correct expression for the Continuous Law of Total Probability?

○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X=x)\,dx$
○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \cap X=x) f_X(x)\,dx$
○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(X=x \mid A)\mathbb{P}(A)\,dx$
○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X=x) f_X(x)\,dx$

2 Joint PMF's

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

a) Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).

b) Find the marginal PMF for X, $p_X(x)$ for $x \in \Omega_X$.

c) Find the marginal PMF for Y, $p_Y(y)$ for $y \in \Omega_Y$.

d) Are X and Y independent? Why or why not?

e) Find $\mathbb{E}\left[X^3 Y\right]$.

3 Trinomial Distribution

A generalization of the Binomial model is when there is a sequence of n independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let X_i be the number of times outcome i occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$.

4 Do You “Urn” to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the i-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

a) X_1, X_2

b) X_1, X_2, X_3

5 Successes

Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first 2 successes. Find the joint pmf of X_1 and X_2. Write an expression for $E[\sqrt{X_1 X_2}]$. You can leave your answer in the form of a sum.

6 Continuous joint density

The joint density of X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of W and V is given by

$$f_{W,V}(w, v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent? Are W and V independent?

7 3 points on a line

Three values X_1, X_2, X_3 are selected uniformly at random, each between 0 and 1 (continuous independent uniform distributions). What is the probability that X_2 is greater than X_1 but less than X_3?

8 Min and max of i.i.d. random variables

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables each with CDF $F_X(x)$ and pdf $f_X(x)$. Let $Y = \min(X_1, \ldots, X_n)$ and let $Z = \max(X_1, \ldots, X_n)$. Show how to write the CDF and pdf of Y and Z in terms of the functions $F_X(\cdot)$ and $f_X(\cdot)$.

9 Law of Total Probability

a) Suppose we flip a coin with probability U of heads, where U is equally likely to be one of $\Omega_U = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ (notice this set has size $n+1$). Let H be the event that the coin comes up heads. What is $\mathbb{P}(H)$?

b) Now suppose $U \sim \mathsf{Uniform(0,1)}$ has the *continuous* uniform distribution over the interval $[0, 1]$. Use the continuous law of total probability to handle this case.

c) Suppose that X_1 and X_2 are independent continuous random variables. Find an expression for $\mathbb{P}(X_1 < 2X_2)$ using the law of total probability, in terms of $F_{X_1}, F_{X_2}, f_{X_1}, f_{X_2}$. (Your answer will be in the form of a single integral, and requires no calculations – do not evaluate it).

d) Suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Find s, where $\Phi(s) = \mathbb{P}(X_1 < 2X_2)$ using the fact that linear combinations of independent normal random variables are still normal.

e) Suppose $Z = X + Y$, where X and Y are independent. Z is called the *convolution* of the two random variables. If X, Y, Z are discrete, using the law of total probability, we can write

$$p_Z(z) = \mathbb{P}(X + Y = z) = \sum_x \mathbb{P}(X = x \cap Y = z - x) = \sum_x p_X(x)\, p_Y(z - x)$$

Write an analogous expression for $F_Z(z)$ in the case that X, Y, Z are continuous where, again, X and Y are independent.

10 Jointly distributed random variables involving 3 variables

a) **Validating a Joint Density.** Let X, Y, and Z be continuous random variables. To verify that $f_{X,Y,Z}(x, y, z) = 6$ for the region $0 \le x \le y \le z \le 1$ (and 0 otherwise) is a valid joint probability density function, which of the following equations must hold true?

○ $\int_0^1 \int_0^1 \int_0^1 6\, dx\, dy\, dz = 1$

○ $\int_0^1 \int_0^z \int_0^y 6\, dx\, dy\, dz = 1$

○ $\int_0^1 \int_0^x \int_0^y 6\, dz\, dy\, dx = 1$

○ $\int_0^1 \int_x^1 \int_y^1 6\,dx\,dy\,dz = 1$

b) **Integrating out a Variable.** Let X, Y, and Z be continuous random variables with joint PDF $f_{X,Y,Z}(x,y,z) = 6$ for the region $0 \leq x \leq y \leq z \leq 1$, and 0 otherwise. Which of the following correctly expresses the joint marginal PDF $f_{X,Y}(x,y)$ for the valid region?

○ $\int_0^1 6\,dz$

○ $\int_x^y 6\,dz$

○ $\int_0^y 6\,dz$

○ $\int_y^1 6\,dz$

c) **The 3D Simplex.** Let X, Y, and Z be independent random variables, each uniformly distributed over $(0,1)$. Which of the following integrals correctly computes the probability that $X + Y + Z \leq 1$?

○ $\int_0^1 \int_0^1 \int_0^1 1\,dz\,dy\,dx$

○ $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx$

○ $\int_0^1 \int_0^x \int_0^y 1\,dz\,dy\,dx$

○ $\int_0^1 \int_0^{1-x} \int_0^1 1\,dz\,dy\,dx$

d) **Bounding with Max.** Let X, Y, and Z be independent random variables, each uniformly distributed over $(0,1)$. Which of the following integrals correctly computes $\mathbb{P}(X \geq \max(Y,Z))$?

○ $\int_0^1 \int_x^1 \int_x^1 1\,dz\,dy\,dx$

○ $\int_0^1 \int_0^1 \int_0^{\max(y,z)} 1\,dx\,dy\,dz$

○ $\int_0^1 \int_0^x \int_0^x 1\,dz\,dy\,dx$

○ $\int_0^1 \int_0^1 \int_{\min(y,z)}^1 1\,dx\,dy\,dz$

e) **Conditional PDF.** (Not covered in class.) For two continuous random variables X and Y, which of the following defines the conditional PDF $f_{X|Y}(x|y)$?

○ $\frac{f_{X,Y}(x,y)}{f_X(x)}$

○ $\frac{f_{X,Y}(x,y)}{f_Y(y)}$

○ $f_X(x) f_Y(y)$

○ $\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$