CSE 312 – Section 8 Solutions

Spring 2026

Review of Main Concepts

- **Multivariate: Discrete to Continuous:**

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x, s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Independence must have	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

- **Law of Total Probability (r.v. version):** If X is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A | X = x) p_X(x) \qquad \text{discrete } X$$

- **Continuous Law of Total Probability:**

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A | X = x) f_X(x) dx$$

There will be problems covering the following concepts (some of which have not yet been discussed in lecture) on the Section 9 worksheet:

- **Conditional expectation:** The expected value of random variable X given that event A has occurred, written $\mathbb{E}[X|A]$, is defined as

$$\mathbb{E}[X|A] = \sum_{x \in \Omega_X} x \cdot \mathbb{P}(X = x | A).$$

- **Discrete Law of Total Expectation (event version):** Let $A_1 \dots, A_n$ be a partition of the sample space. Then

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X | A_i] \mathbb{P}(A_i).$$

- **Discrete Law of Total Expectation (r.v. version):** Let X and Y be two random variables. Then
$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X|Y=y] \cdot \mathbb{P}(Y=y).$$

- **Continuous Law of Total Expectation:**
$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X|Y=y] f_Y(y) dy$$

- **Expected value of X conditioned on r.v. Y:** Suppose that Y is a random variable that takes values $y_1, \ldots, y_k$. Then $\mathbb{E}[X|Y]$ is the following random variable
$$\mathbb{E}[X|Y] = \begin{cases} \mathbb{E}[X|Y=y_1] & \text{with probability } \mathbb{P}(Y=y_1) \\ \mathbb{E}[X|Y=y_2] & \text{with probability } \mathbb{P}(Y=y_2) \\ \ldots & \\ \mathbb{E}[X|Y=y_k] & \text{with probability } \mathbb{P}(Y=y_k) \end{cases}$$

- **Law of total expectation (rewritten):** Given the above definition, we can write
$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \sum_{i=1}^{k} \mathbb{E}[X|Y=y_i] \cdot \mathbb{P}(Y=y_i).$$

- **Covariance:** We may not get to this in class, but there is a problem on the pset about it. To find out more, check out section 5.4 in the Tsun book. And now the definition: For any two random variables X, Y the *covariance* is defined as
$$\text{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$
It can also be shown that
$$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

- **Conditional distributions:** We are not explicitly covering this topic in class, but it is **highly** recommended that you study it. Much of the above can be more appropriately rewritten in terms of conditional distributions. See Tsun, Section 5.3.

	Discrete	**Continuous**
Conditional PMF/PDF	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X\|Y=y] = \sum_x x p_{X\|Y}(x\|y)$	$\mathbb{E}[X\|Y=y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$

Plan for Section

- Content Review (Problem 1)
- Joint PMF's (Problem 2) - do it fast
- Continuous joint density - Problem 6

- 3 points on a line - Problem 7
- Min and max of i.i.d. random variables - Problem 8 if time permits

We recommend that students look at the final problem for examples of how to set up the ranges of integration. Might be helpful on the homework.

1 Content Review

a) Select one: Given two discrete random variables X and Y, the joint CDF is

○ $F_{X,Y}(x, y) = \sum_{t<x} p_{X,Y}(t, y)$
○ $F_{X,Y}(x, y) = \sum_{s<y} p_{X,Y}(x, s)$
○ $F_{X,Y}(x, y) = \sum_{t\leq x} \sum_{s\leq y} p_{X,Y}(t, s)$
○ $F_{X,Y}(x, y) = p_{X,Y}(x, y)$

> The third answer follows directly from the definition of multivariate / joint distributions.

b) **Marginal PDF.** Let X and Y be continuous random variables with joint PDF $f_{X,Y}(x, y)$. Which of the following correctly expresses the marginal PDF $f_X(x)$?

○ $\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx$
○ $\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
○ $\frac{f_{X,Y}(x,y)}{f_Y(y)}$
○ $\int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t, s)\, ds\, dt$

> **Answer:** $\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$
> To find the marginal distribution of X, you must "integrate out" the other variable, Y, over its entire range. *(Note: The first option incorrectly integrates out x, which would leave a function of y. The third option is the conditional PDF $f_{X|Y}(x|y)$, and the fourth option is the joint CDF $F_{X,Y}(x, y)$.)*

c) **Independence and Support.** True or False: If the joint support $\Omega_{X,Y}$ of the random variables (X, Y) is a circle defined by $x^2 + y^2 \leq 1$, and $\Omega_X = \Omega_Y = [0, 1]$ then X and Y are independent.

○ True
○ False

> **Answer: False**
> For X and Y to be independent, their joint support must be the Cartesian product of their marginal supports ($\Omega_{X,Y} = \Omega_X \times \Omega_Y$). A circle is not the Cartesian product of

$[0, 1]$.

d) **Continuous Law of Total Probability.** Let A be an event and X be a continuous random variable with PDF $f_X(x)$. Which of the following is the correct expression for the Continuous Law of Total Probability?

- ○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X = x)\, dx$
- ○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \cap X = x) f_X(x)\, dx$
- ○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(X = x \mid A)\mathbb{P}(A)\, dx$
- ○ $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X = x) f_X(x)\, dx$

Answer: $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid X = x) f_X(x)\, dx$
The Continuous Law of Total Probability scales the conditional probability of A given $X = x$ by the density of X at x, integrated over all possible values of X. *(Note: Option 1 is missing the density weighting. Option 2 incorrectly combines an intersection with the density function. Option 3 is a jumbled application of Bayes' rule elements.)*

2 Joint PMF's

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

a) Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).

$\Omega_X = \{0, 1\}$, $\Omega_Y = \{1, 2, 3\}$, and $\Omega_{X,Y} = \{(0, 2), (0, 3), (1, 1), (1, 3)\}$

b) Find the marginal PMF for X, $p_X(x)$ for $x \in \Omega_X$.

$$p_X(0) = \sum_y p_{X,Y}(0, y) = 0 + 0.2 + 0.1 = 0.3$$

$$p_X(1) = 1 - p_X(0) = 0.7$$

c) Find the marginal PMF for Y, $p_Y(y)$ for $y \in \Omega_Y$.

$$p_Y(1) = \sum_x p_{X,Y}(x, 1) = 0 + 0.3 = 0.3$$

$$p_Y(2) = \sum_x p_{X,Y}(x, 2) = 0.2 + 0 = 0.2$$

$$p_Y(3) = \sum_x p_{X,Y}(x,3) = 0.1 + 0.4 = 0.5$$

d) Are X and Y independent? Why or why not?

No, since a necessary condition is that $\Omega_{X,Y} = \Omega_X \times \Omega_Y$.

e) Find $\mathbb{E}\left[X^3 Y\right]$.

Note that $X^3 = X$ since X takes values in $\{0, 1\}$.

$$\mathbb{E}\left[X^3 Y\right] = \mathbb{E}\left[XY\right] = \sum_{(x,y)\in\Omega_{X,Y}} xy p_{X,Y}(x,y) = 1 \cdot 1 \cdot 0.3 + 1 \cdot 3 \cdot 0.4 = 1.5$$

3 Trinomial Distribution

A generalization of the Binomial model is when there is a sequence of n independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let X_i be the number of times outcome i occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$.

We use a similar argument as for the binomial PMF. $\binom{n}{x_1,x_2,x_3}$ is the number of ways to select which of the n outcomes result in each of the 3 outcomes. Then, we multiply the probabilities of each trial being the corresponding outcome (e.g., $p_1^{x_1}$ is the probability that all x_1 trials end up being outcome 1). This gives use the following PMF:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} \prod_{i=1}^{3} p_i^{x_i} = \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

where $x_1 + x_2 + x_3 = n$ and are nonnegative integers.

4 Do You "Urn" to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the i-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

a) X_1, X_2

Here is one way of defining the joint pmf of X_1, X_2

$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156}$$

$$\mathbb{P}(X_1 = 1, X_2 = 0) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 0 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{8}{12} = \frac{40}{156}$$
$$\mathbb{P}(X_1 = 0, X_2 = 1) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 1 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{5}{12} = \frac{40}{156}$$
$$\mathbb{P}(X_1 = 0, X_2 = 0) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{56}{156}$$

b) X_1, X_2, X_3

Instead of listing out all the individual probabilities, we could write a more compact formula for the pmf. In this problem, the denominator is always $P(13, k)$, where k is the number of random variables in the joint pmf. And the numerator is $P(5, i)$ times $P(8, j)$ where i and j are the number of 1s and 0s, respectively. If we wish to compute $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$, then the number of 1s (i.e., white balls) is $x_1 + x_2 + x_3$, and the number of 0s (i.e., red balls) is $(1 - x_1) + (1 - x_2) + (1 - x_3)$. Then, we can write the pmf as follows:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{10!}{13!} \cdot \frac{5!}{(5 - x_1 - x_2 - x_3)!} \cdot \frac{8!}{(5 + x_1 + x_2 + x_3)!}$$

5 Successes

Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first 2 successes. Find the joint pmf of X_1 and X_2. Write an expression for $E[\sqrt{X_1 X_2}]$. You can leave your answer in the form of a sum.

X_1 and X_2 take on two particular values x_1 and x_2, when there are x_1 failures followed by one success, and then x_2 failures followed by one success. Since the Bernoulli trials are independent the joint pmf is

$$p_{X_1,X_2}(x_1, x_2) = (1 - p)^{x_1} p \cdot (1 - p)^{x_2} p = (1 - p)^{x_1+x_2} p^2$$

for $(x_1, x_2) \in \Omega_{X_1,X_2} = \{0, 1, 2, ...\} \times \{0, 1, 2, ...\}$ By the definition of expectation

$$E[\sqrt{X_1 X_2}] = \sum_{(x_1,x_2) \in \Omega_{X_1,X_2}} \sqrt{x_1 x_2} \cdot (1 - p)^{x_1+x_2} p^2.$$

6 Continuous joint density

The joint density of X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of W and V is given by

$$f_{W,V}(w,v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent? Are W and V independent?

For two random variables X, Y to be independent, we must have $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x \in \Omega_X,\ y \in \Omega_Y$. Let's start with X and Y by finding their marginal PDFs. By definition, and using the fact that the joint PDF is 0 outside of $y > 0$, we get:

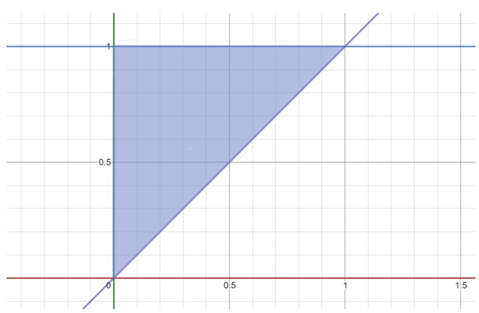
$$f_X(x) = \int_0^\infty xe^{-(x+y)}dy = e^{-x}x$$

We do the same to get the PDF of Y, again over the range $x > 0$:

$$f_Y(y) = \int_0^\infty xe^{-(x+y)}dx = e^{-y}$$

Since $e^{-x}x \cdot e^{-y} = xe^{-x-y} = xe^{-(x+y)}$ for all $x, y > 0$, X and Y are independent.

We can see that W and V are not independent simply by observing that $\Omega_W = (0,1)$ and $\Omega_V = (0,1)$, but $\Omega_{W,V}$ is not equal to their Cartesian product. Specifically, looking at their range of $f_{W,V}(w,v)$. Graphing it with w as the "x-axis" and v as the "y-axis", we see that :



The shaded area is where the joint pdf is strictly positive. Looking at it, we can see that it is not rectangular, and therefore it is not the case that $\Omega_{W,V} = \Omega_W \times \Omega_V$. Remember, the joint range being the Cartesian product of the marginal ranges is not sufficient for independence, but it is *necessary*. Therefore, this is enough to show that they are not independent.

7 3 points on a line

Three values X_1, X_2, X_3 are selected uniformly at random, each between 0 and 1 (continuous independent uniform distributions). What is the probability that X_2 is greater than X_1 but less than X_3?

There are mutliple ways to approach this problem! Let $X_1, X_2, X_3 \sim Unif(0, 1)$. We want to find $\mathbb{P}(X_1 < X_2 < X_3)$.

Method 1: Condition on X_2.

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < X_2 < X_3 \mid X_2 = x)\, f_{X_2}(x)\, dx && \text{Continuous LoTP} \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x, X_3 > x | X_2 = x)\, f_{X_2}(x)\, dx \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x, X_3 > x)\, f_{X_2}(x)\, dx && \text{Indep of } X_1, X_3 \text{ from } X_2 \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x)\, \mathbb{P}(x < X_3)\, f_{X_2}(x)\, dx && \text{Indep of } X_1, X_3 \\
&= \int_{-\infty}^{\infty} F_{X_1}(x)\, (1 - F_{X_3}(x))\, f_{X_2}(x)\, dx \\
&= \int_{0}^{1} x\, (1 - x)\, 1\, dx \\
&= \frac{x^2}{2} - \frac{x^3}{3} \bigg|_0^1 = \frac{1}{6}
\end{aligned}$$

Method 2: Integrate over each variable. First, find the joint PDF of the three variables using independence

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) = \begin{cases} 1 & \text{if} \quad 0 \le x_1, x_2, x_3 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

So, the probability of the event $X_1 < X_2 < X_3$ is exactly the volume of the region where $x_1 < x_2 < x_2$. In order to integrate over these region, start with x_2 (this has a range of 0 to 1), then x_1 (since $x_1 < x_2$, its range is 0 to x_2), and finally x_3 (since $x_2 < x_3 < 1$, its range is x_2 to 1).

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \int_0^1 \int_0^{x_2} \int_{x_2}^1 f_{X_1, X_2, X_3}(X_1, X_2, X_3)\, dx_3 dx_1 dx_2 \\
&= \int_0^1 \int_0^{x_2} \int_{x_2}^1 1\, dx_3 dx_1 dx_2 \\
&= \int_0^1 \int_0^{x_2} x_3 \bigg|_{x_2}^1 dx_1 dx_2 = \int_0^1 \int_0^{x_2} 1 - x_2\, dx_1 dx_2 \\
&= \int_0^1 (1 - x_2) x_1 \bigg|_0^{x_2} dx_2 = \int_0^1 (1 - x_2) x_2\, dx_2 \\
&= \frac{x^2}{2} - \frac{x^3}{2} \bigg|_0^1 = \frac{1}{6}
\end{aligned}$$

Method 3: Identify symmetry. The key idea is to note that since X_1, X_2, X_3 are i.i.d, every ordered sequence is equally likely, i.e $P(X_1 < X_2 < X_3) = P(X_1 < X_3 < X_2) = \cdots = P(X_3 < X_2 < X_1)$.

Since there are 3 variables, there are $3! = 6$ equally likely sequences. Also note that the 6 events are disjoint (no set of them can be occur simultaneously) and it is impossible for none of them to occur (ties like $X_1 = X_2$ have probability 0). Therefore, they partition the sample space, so their probabilities should add up to 1.

$$\begin{aligned}
1 &= P(X_1 < X_2 < X_3) + \cdots + P(X_3 < X_2 < X_1) && \text{Events partition the sample space}\\
&= 6 \cdot P(X_1 < X_2 < X_3) && \text{Equally likely sequences}\\
&\implies P(X_1 < X_2 < X_3) = \frac{1}{6}
\end{aligned}$$

8 Min and max of i.i.d. random variables

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables each with CDF $F_X(x)$ and pdf $f_X(x)$. Let $Y = \min(X_1, \ldots, X_n)$ and let $Z = \max(X_1, \ldots, X_n)$. Show how to write the CDF and pdf of Y and Z in terms of the functions $F_X(\cdot)$ and $f_X(\cdot)$.

First we compute the CDFs of Z and Y as follows:

$$\begin{aligned}
F_Z(z) &= P(Z < z)\\
&= P(X_1 < z, \ldots, X_n < z) && \text{[Definition of max]}\\
&= P(X_1 < z) \cdot \ldots \cdot P(X_n < z) && \text{[Independence]}\\
&= (F_X(z))^n
\end{aligned}$$

$$\begin{aligned}
F_Y(y) &= P(Y < y)\\
&= 1 - P(Y > y)\\
&= 1 - P(X_1 > y, \ldots, X_n > y) && \text{[Definition of min]}\\
&= 1 - P(X_1 > y) \cdot \ldots \cdot P(X_n > y) && \text{[Independence]}\\
&= 1 - (1 - F_X(y))^n
\end{aligned}$$

Using the fact that $f_X(x) = \frac{d}{dx} F_X(x)$ and the CDFs that we found we can compute the pdfs of Z and Y as follows:

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z)\\
&= \frac{d}{dz} (F_X(z))^n\\
&= n \cdot F_X(z)^{n-1} \cdot \left(\frac{d}{dz} F_X(z)\right)\\
&= n \cdot F_X(z)^{n-1} \cdot f_X(z)
\end{aligned}$$

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{d}{dy}\left(1-(1-F_X(y))^n\right) \\
&= -n \cdot (1-F_X(y))^{n-1} \cdot \frac{d}{dy}(1-F_X(y)) \\
&= n \cdot (1-F_X(y))^{n-1} \cdot f_X(y)
\end{aligned}
$$

9 Law of Total Probability

a) Suppose we flip a coin with probability U of heads, where U is equally likely to be one of $\Omega_U = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$ (notice this set has size $n+1$). Let H be the event that the coin comes up heads. What is $\mathbb{P}(H)$?

We can use the law of total probability, conditioning on $U = \frac{k}{n}$ for $k = 0, ..., n$.

$$
\begin{aligned}
\mathbb{P}(H) = \sum_{k=0}^{n} \mathbb{P}\left(H \mid U = \frac{k}{n}\right) \mathbb{P}\left(U = \frac{k}{n}\right) = \sum_{k=0}^{n} \frac{k}{n} \cdot \frac{1}{n+1} \\
= \frac{1}{n(n+1)} \sum_{k=0}^{n} k = \frac{1}{n(n+1)} \frac{n(n+1)}{2} = \frac{1}{2}
\end{aligned}
$$

b) Now suppose $U \sim \mathsf{Uniform(0,1)}$ has the *continuous* uniform distribution over the interval $[0,1]$. Use the continuous law of total probability to handle this case.

We can perform basically the same process as above, just using an integral instead of a sum. The values that U can take on are anywhere in the continuous interval $[0,1]$, so we integrate over that with respect to u and use the PDF of U. Plugging that in we can get the same answer of $\frac{1}{2}$ as before.

$$
\mathbb{P}(H) = \int_0^1 \mathbb{P}(H \mid U = u) f_U(u) du = \int_0^1 u \cdot 1 du = \frac{1}{2}[u^2]_0^1 = \frac{1}{2}
$$

c) Suppose that X_1 and X_2 are independent continuous random variables. Find an expression for $\mathbb{P}(X_1 < 2X_2)$ using the law of total probability, in terms of $F_{X_1}, F_{X_2}, f_{X_1}, f_{X_2}$. (Your answer will be in the form of a single integral, and requires no calculations – do not evaluate it).

We use the continuous version of the "Law of Total Probability" to integrate over all possible values of X_2:

$$
\mathbb{P}(X_1 < 2X_2) = \int_{-\infty}^{\infty} \mathbb{P}(X_1 < 2X_2 \mid X_2 = x_2) f_{X_2}(x_2) dx_2 = \int_{-\infty}^{\infty} F_{X_1}(2x_2)\, f_{X_2}(x_2)\, dx_2
$$

d) Suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$. Find s, where $\Phi(s) = \mathbb{P}(X_1 < 2X_2)$ using the fact that linear combinations of independent normal random variables are still normal.

Let $X_3 = X_1 - 2X_2$, so that $X_3 \sim \mathcal{N}(\mu_1 - 2\mu_2, \sigma_1^2 + 4\sigma_2^2)$ (by the reproductive property of normal distributions)

$$\mathbb{P}(X_1 < 2X_2) = \mathbb{P}(X_1 - 2X_2 < 0) = \mathbb{P}(X_3 < 0) = \mathbb{P}\left(\frac{X_3 - (\mu_1 - 2\mu_2)}{\sqrt{\sigma_1^2 + 4\sigma_2^2}} < \frac{0 - (\mu_1 - 2\mu_2)}{\sqrt{\sigma_1^2 + 4\sigma_2^2}}\right)$$

$$= \mathbb{P}\left(Z < \frac{2\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + 4\sigma_2^2}}\right) = \Phi\left(\frac{2\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + 4\sigma_2^2}}\right) \implies s = \frac{2\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + 4\sigma_2^2}}$$

e) Suppose $Z = X + Y$, where X and Y are independent. Z is called the *convolution* of the two random variables. If X, Y, Z are discrete, using the law of total probability, we can write

$$p_Z(z) = \mathbb{P}(X + Y = z) = \sum_x \mathbb{P}(X = x \cap Y = z - x) = \sum_x p_X(x)\, p_Y(z - x)$$

Write an analogous expression for $F_Z(z)$ in the case that X, Y, Z are continuous where, again, X and Y are independent.

$$F_Z(z) = \mathbb{P}(X + Y \le z) = \int_{-\infty}^{\infty} \mathbb{P}(Y \le z - X \mid X = x) f_X(x) dx = \int_{-\infty}^{\infty} F_Y(z - x) f_X(x) dx$$

10 Jointly distributed random variables involving 3 variables

a) **Validating a Joint Density.** Let X, Y, and Z be continuous random variables. To verify that $f_{X,Y,Z}(x, y, z) = 6$ for the region $0 \le x \le y \le z \le 1$ (and 0 otherwise) is a valid joint probability density function, which of the following equations must hold true?

- ○ $\int_0^1 \int_0^1 \int_0^1 6\, dx\, dy\, dz = 1$
- ○ $\int_0^1 \int_0^z \int_0^y 6\, dx\, dy\, dz = 1$
- ○ $\int_0^1 \int_0^x \int_0^y 6\, dz\, dy\, dx = 1$
- ○ $\int_0^1 \int_x^1 \int_y^1 6\, dx\, dy\, dz = 1$

Answer: $\int_0^1 \int_0^z \int_0^y 6\, dx\, dy\, dz = 1$
To be a valid joint PDF, the integral of the density over the entire valid region must equal 1. The support is given by the chain of inequalities $0 \le x \le y \le z \le 1$. Setting up the bounds from the outside in using the order $dx\, dy\, dz$: the outermost variable z ranges from absolute minimum to maximum (0 to 1). For a fixed z, the variable y is bounded between 0 and z. Finally, for fixed y and z, the innermost variable x is

bounded between 0 and y.

$$\int_0^1 \int_0^z \int_0^y 6\,dx\,dy\,dz = 1$$

b) **Integrating out a Variable.** Let X, Y, and Z be continuous random variables with joint PDF $f_{X,Y,Z}(x,y,z) = 6$ for the region $0 \le x \le y \le z \le 1$, and 0 otherwise. Which of the following correctly expresses the joint marginal PDF $f_{X,Y}(x,y)$ for the valid region?

- ○ $\int_0^1 6\,dz$
- ○ $\int_x^y 6\,dz$
- ○ $\int_0^y 6\,dz$
- ○ $\int_y^1 6\,dz$

Answer: $\int_y^1 6\,dz$
To find the joint marginal PDF $f_{X,Y}(x,y)$, we must "integrate out" the variable Z over its valid range. Given the support bounds $0 \le x \le y \le z \le 1$, for any fixed values of x and y, the variable z ranges from a minimum of y to a maximum of 1.

$$f_{X,Y}(x,y) = \int_y^1 f_{X,Y,Z}(x,y,z)\,dz = \int_y^1 6\,dz$$

(Note: The bounds are the key trap here. Because the joint PDF is only non-zero when $z \ge y$, the lower limit of integration must be y, not 0 or x.)

c) **The 3D Simplex.** Let X, Y, and Z be independent random variables, each uniformly distributed over $(0,1)$. Which of the following integrals correctly computes the probability that $X + Y + Z \le 1$?

- ○ $\int_0^1 \int_0^1 \int_0^1 1\,dz\,dy\,dx$
- ○ $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx$
- ○ $\int_0^1 \int_0^x \int_0^y 1\,dz\,dy\,dx$
- ○ $\int_0^1 \int_0^{1-x} \int_0^1 1\,dz\,dy\,dx$

Answer: $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx$
First, the joint density of X, Y, Z is $f_{X,Y,Z}(x,y,z) = 1$ due to independence and standard uniform distributions. We need to integrate this density over the region where $X+Y+Z \le 1$. If we fix the outer bounds for $X = x$ from 0 to 1, the remaining "budget" for Y is $1-x$, so Y integrates from 0 to $1-x$. Finally, given $X = x$ and $Y = y$, the

variable Z is bounded above by $1-x-y$.

$$\mathbb{P}(X+Y+Z \leq 1) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\, dz\, dy\, dx$$

d) **Bounding with Max.** Let X, Y, and Z be independent random variables, each uniformly distributed over $(0,1)$. Which of the following integrals correctly computes $\mathbb{P}(X \geq \max(Y,Z))$?

- ○ $\int_0^1 \int_x^1 \int_x^1 1\, dz\, dy\, dx$
- ○ $\int_0^1 \int_0^1 \int_0^{\max(y,z)} 1\, dx\, dy\, dz$
- ○ $\int_0^1 \int_0^x \int_0^x 1\, dz\, dy\, dx$
- ○ $\int_0^1 \int_0^1 \int_{\min(y,z)}^1 1\, dx\, dy\, dz$

Answer: $\int_0^1 \int_0^x \int_0^x 1\, dz\, dy\, dx$
The joint density is $f_{X,Y,Z}(x,y,z) = 1$. The condition $X \geq \max(Y,Z)$ is equivalent to stating that $X \geq Y$ **and** $X \geq Z$. It is easiest to set x as the outermost integral bounding from 0 to 1. Then, for any fixed value of $X = x$, both Y and Z are independently bounded to be less than or equal to x. Thus, y ranges from 0 to x, and z ranges from 0 to x.

$$\mathbb{P}(X \geq \max(Y,Z)) = \int_0^1 \int_0^x \int_0^x 1\, dz\, dy\, dx$$

e) **Conditional PDF.** (Not covered in class.) For two continuous random variables X and Y, which of the following defines the conditional PDF $f_{X|Y}(x|y)$?

- ○ $\frac{f_{X,Y}(x,y)}{f_X(x)}$
- ○ $\frac{f_{X,Y}(x,y)}{f_Y(y)}$
- ○ $f_X(x) f_Y(y)$
- ○ $\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

Answer: $\frac{f_{X,Y}(x,y)}{f_Y(y)}$
The conditional PDF of X given $Y = y$ is the joint PDF divided by the marginal PDF of Y evaluated at y. *(Note: Option 1 is $f_{Y|X}(y|x)$, option 3 is the joint PDF only if X and Y are independent, and option 4 is the marginal PDF of X.)*