**Student name:** ______________ **Student Number:** ______________

CSE 312: Foundations of Computing II - Section A Section B Spring 2022

# Midterm Exam

**Important: Do not read the following pages until instructed to start. Read the instructions on this page carefully.**

**Instructions.** This midterm is meant to be solved in 50 minutes. **The start time is 9:30am, 1:30pm, and you are required to stop writing at 10:20am, 2:20pm,** unless otherwise instructed. When done, wait for a proctor to collect your solution, or follow any other instructions by the proctors.

- This exam consists of **five tasks**, overall with **100 points**.
- **Write your name and student number on top of this page**
- This is a **closed-book exam.** No written document is allowed. In particular, you are not allowed to use any textbooks, nor additional personal notes, homework assignments or solutions, etc.
- A detachable cheat sheet is provided on the front and back of the last page of this midterm. Please be gentle when removing it.
- **No electronics are allowed** during the exam (no smart phones, no laptops, no smart watches, no pocket calculators, etc). Keep them stored in your bag.
- Write your solutions in the appropriate spaces (additional empty space is available at the end of the midterm – please reference to it if you use this space).

## Good luck!

## Task 1 – Short Questions [20 pts]

Are the following statements True or False? Provide a short justification for your answer, explaining why the statement is true or false.

**a)** $\sum_{k=0}^{n} \binom{n}{k} 3^k = 4^n$.

**b)** If $A$ and $B$ are events then $\mathbb{P}(A) \leqslant \mathbb{P}(A \cap B) + \mathbb{P}(B^c)$.

**c)** Let $A$ and $B$ be events such that $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.3$ and $\mathbb{P}(A \cup B) = 0.65$. Are the events $A$ and $B$ mutually independent?

**d)** If $X$ is an indicator variable such that $\mathbb{P}(X = 1) = 0.25$, then $\mathbb{E}\left[X^2\right] = \mathbb{E}[X]^2$.

**e)** If $X$ and $Y$ are independent random variables, then $\mathbb{E}\left[(X + Y)^2\right] = \mathbb{E}\left[X^2\right] + 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}\left[Y^2\right]$.

## Task 2 – Counting [15 pts]

For both tasks, your final answers can contain factorials and/or binomial coefficients. However, do evaluate small factorials (up to $3! = 6$).

**a)** How many ways are there to re-order the 26 alphabet letters A, B, ..., Z so that A is not next to B.

**b)** How many integer solutions $(x_1, x_2, x_3)$ to the equation $x_1 + x_2 + x_3 = 12$ are there such that $x_1 \geqslant 1$, $x_2 \geqslant 2$, and $x_3 \geqslant 3$

## Task 3 – Basic Probability [20 pts]

A jar contains 4 red balls numbered 1 to 4 and 7 white balls numbered 1 to 7. We choose three balls from the jar, one after another, uniformly at random, without replacing any after each choice.

**a)** What is the probability that the first and third balls chosen have the same number?

**b)** What is the probability that the three balls we chose contain two with the same number?

**c)** What is the probability that we have chosen balls of both colors?

A jar contains 5 red balls numbered 1 to 5 and 6 white balls numbered 1 to 6. We choose three balls from the jar, one after another, uniformly at random, without replacing any after each choice.

**a)** What is the probability that the first and third balls chosen have the same number?

**b)** What is the probability that the three balls we chose contain two with the same number?

**c)** What is the probability that we have chosen balls of both colors?

## Task 4 – Conditional Probabilities [20 pts]

Jordan has a standard fair 6-sided die and three cards. The first card is red on both sides. The first card is blue on both sides. The second card is red on one side, and blue on the other. The third card is blue on both sides. The third card is red on both sides.

Jordan rolls the die to choose one of the three cards.

- If he rolls a 1, he selects the first card
- If he rolls a 2 or a 3, he selects the second card
- If he rolls a 4, 5, or 6, he selects the third card.

He then shows you a random side of the card he has chosen (each side equally likely).

**a)** What is the probability that you see a red side?

**b)** If you see a red side, what is the probability that the other side of the card is blue?

## Task 5 – Luggage swap [25 pts]

Alex, Barbara, and Carol have suitcases that look identical to each other but different from the suitcases of all the other passengers on their flight. At baggage claim they couldn't tell each other's suitcases apart and, one after another, each chose a uniformly random suitcase from the ones that were left that looked like theirs, so the suitcases were chosen in a uniformly random order.
Let $X$ be the number of people from $\{$Alex, Barbara, Carol$\}$ who ended up with the wrong suitcase.

**a)** Give the PMF of $X$. (In particular, give the range $\Omega_X$ of $X$.)

**b)** Compute $\mathbb{E}[X]$ directly using **a)**.

**c)** Compute $\mathbb{E}[X]$ using the linearity of expectation and indicator random variables.

**d)** What is $\mathbb{E}\left[X^2\right]$? Simplify the result as far as possible.

**e)** What is $\text{Var}(X)$? (Try to compute this as efficiently as possible.)

## Task 6 – Safety Cards [25 pts]

Alex, Barbara, and Carol are on a game show. On the show, a dealer shuffles 3 cards labeled A, B, C and randomly and uniformly deals one card to each of Alex, Barbara, and Carol. If the player gets a card with a letter that begins their first name, they are safe, otherwise they are "at risk" and must answer a skill-testing question.
Let $X$ be the number of contestants among Alex, Barbara, and Carol who are at risk.

**a)** Give the PMF of $X$. (In particular, give the range $\Omega_X$ of $X$.)

**b)** Compute $\mathbb{E}[X]$ directly using **a)**.

**c)** Compute $\mathbb{E}[X]$ using the linearity of expectation and indicator random variables.

**d)** What is $\mathbb{E}[X^2]$? Simplify the result as far as possible.

**e)** What is $\text{Var}(X)$? (Try to compute this as efficiently as possible.)

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