Student name: ________________ **Student Number:** ________________

CSE 312: Foundations of Computing II - Section A Section B Spring 2022

Midterm Exam

Solutions

Important: Do not read the following pages until instructed to start. Read the instructions on this page carefully.

Instructions. This midterm is meant to be solved in 50 minutes. **The start time is 9:30am, 1:30pm, and you are required to stop writing at 10:20am, 2:20pm,** unless otherwise instructed. When done, wait for a proctor to collect your solution, or follow any other instructions by the proctors.

- This exam consists of **five tasks**, overall with **100 points**.
- **Write your name and student number on top of this page**
- This is a **closed-book exam.** No written document is allowed. In particular, you are not allowed to use any textbooks, nor additional personal notes, homework assignments or solutions, etc.
- A detachable cheat sheet is provided on the front and back of the last page of this midterm. Please be gentle when removing it.
- **No electronics are allowed** during the exam (no smart phones, no laptops, no smart watches, no pocket calculators, etc). Keep them stored in your bag.
- Write your solutions in the appropriate spaces (additional empty space is available at the end of the midterm – please reference to it if you use this space).

Good luck!

Task 1 – Short Questions [20 pts]

Are the following statements True or False? Provide a <u>short</u> justification for your answer, explaining why the statement is true or false.

a) $\sum_{k=0}^{n}\binom{n}{k}3^k = 4^n$.

> **True.** The binomial theorem says that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. We can obtain the identity by setting $x = 3$ and $y. = 1$.

b) If A and B are events then $\mathbb{P}(A) \leqslant \mathbb{P}(A \cap B) + \mathbb{P}(B^c)$.

> **True.** Note that B and B^c constitute a partition of the sample space Ω, and therefore, by the LTP, we have
>
> $$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)\ .$$
>
> But then, note that $A \cap B^c \subseteq B^c$, and therefore $\mathbb{P}(A \cap B^c) \leqslant \mathbb{P}(B^c)$.

c) Let A and B be events such that $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.3$ and $\mathbb{P}(A \cup B) = 0.65$. Are the events A and B mutually independent?

> **True.** We can use inclusion-exclusion to compute
>
> $$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.5 + 0.3 - 0.65 = 0.15\ .$$
>
> At the same time, note that $\mathbb{P}(A) \cdot \mathbb{P}(B) = 0.5 \times 0.3 = 0.15$, and thus A, B are independent.

d) If X is an indicator variable such that $\mathbb{P}(X = 1) = 0.25$, then $\mathbb{E}[X^2] = \mathbb{E}[X]^2$.

False. First off, note that for an indicator random variable, $X^2 = X$. Therefore, $\mathbb{E}[X^2] = \mathbb{E}[X] = \mathbb{P}(X = 1) = 1/4$. In contrast, $\mathbb{E}[X]^2 = (1/4)^2 = 1/16 \neq 1/4$.

e) If X and Y are independent random variables, then $\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[Y^2]$.

True. First, note that by linearity of expectation

$$\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2 + 2XY + Y^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] \ .$$

Because X, Y are independent, we also have $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Task 2 – Counting [15 pts]

For both tasks, your final answers can contain factorials and/or binomial coefficients. However, do evaluate small factorials (up to $3! = 6$).

a) How many ways are there to re-order the 26 alphabet letters A, B, ..., Z so that A is not next to B.

We use complementary counting, and count the re-orderings we do not want. We first count the re-orderings which include AB - there are 25 possible positions where AB can occur, i.e., starting with 24 letters to the left, and 0 to the right of AB, all the way to 0 letters to the left, and 24 to the right. As we then need to re-arrange the remaining 24 letters, for which we have $24!$ possibilities, and thus there are $25 \times 24! = 25!$ re-orderings containing AB. Similarly, there are $25!$ re-orderings including BA. Therefore, the overall number of ways to re-order the letters where A is not next to B is

$$26! - 2 \cdot 25! = (26 - 2) \cdot 25! = 24 \cdot 25! \,.$$

b) How many integer solutions (x_1, x_2, x_3) to the equation $x_1 + x_2 + x_3 = 12$ are there such that $x_1 \geqslant 1$, $x_2 \geqslant 2$, and $x_3 \geqslant 3$

This is equivalent to asking for all solutions to the equation $y_1 + y_2 + y_3 = 6$ where $y_1, y_2, y_3 \geqslant 0$. By the encoding/stars and bars method, there are $\binom{8}{2}$ solutions.

Task 3 – Basic Probability [20 pts]

A jar contains 4 red balls numbered 1 to 4 and 7 white balls numbered 1 to 7. We choose three balls from the jar, one after another, uniformly at random, without replacing any after each choice.

a) What is the probability that the first and third balls chosen have the same number?

These two balls are a random subset of two balls of the 11 balls. There are precisely 4 such subsets, each with one red ball and one white ball numbered either 1, 2, 3, or 4. There are $\binom{11}{2}$ such subsets so the probability is

$$\frac{4}{\binom{11}{2}}$$

b) What is the probability that the three balls we chose contain two with the same number?

If we have two balls with the same number then that could be in any one of the 3 pairs of positions. These are disjoint events (since there are only two balls with each number) so the total probability is 3 times the probability for each pair having the same number or

$$\frac{3 \cdot 4}{\binom{11}{2}}.$$

c) What is the probability that we have chosen balls of both colors?

This probability is 1 minus the probability that the balls are all the same color. Among the $\binom{11}{3}$ subsets of 3 balls, there are $\binom{4}{3}$ ways to choose 3 red balls and $\binom{7}{3}$ ways to choose 3 white balls. Therefore this probability is

$$1 - \frac{\binom{4}{3} + \binom{7}{3}}{\binom{11}{3}}$$

A jar contains 5 red balls numbered 1 to 5 and 6 white balls numbered 1 to 6. We choose three balls from the jar, one after another, uniformly at random, without replacing any after each choice.

a) What is the probability that the first and third balls chosen have the same number?

These two balls are a random subset of two balls of the 11 balls. There are precisely 5 such subsets, each with one red ball and one white ball numbered either 1, 2, 3, 4, or 5. There are $\binom{11}{2}$ such subsets so the probability is

$$\frac{5}{\binom{11}{2}}$$

b) What is the probability that the three balls we chose contain two with the same number?

If we have two balls with the same number then that could be in any one of the 3 pairs of positions. These are disjoint events (since there are only two balls with each number) so the total probability is 3 times the probability for each pair having the same number or

$$\frac{3 \cdot 5}{\binom{11}{2}}.$$

c) What is the probability that we have chosen balls of both colors?

This probability is 1 minus the probability that the balls are all the same color. Among the $\binom{11}{3}$ subsets of 3 balls, there are $\binom{5}{3}$ ways to choose 3 red balls and $\binom{6}{3}$ ways to choose 3 white balls. Therefore this probability is

$$1 - \frac{\binom{5}{3} + \binom{6}{3}}{\binom{11}{3}}$$

Task 4 – Conditional Probabilities [20 pts]

Jordan has a standard fair 6-sided die and three cards. The first card is red on both sides. The first card is blue on both sides. The second card is red on one side, and blue on the other. The third card is blue on both sides. The third card is red on both sides.

Jordan rolls the die to choose one of the three cards.

- If he rolls a 1, he selects the first card
- If he rolls a 2 or a 3, he selects the second card
- If he rolls a 4, 5, or 6, he selects the third card.

He then shows you a random side of the card he has chosen (each side equally likely).

a) What is the probability that you see a red side?

Let Red be the event that we see a red card. Let events First, Second, Third be the events that the first, second, and third card is chosen. By the LTP, we have

$$\begin{aligned}\mathbb{P}(\mathrm{Red}) &= \mathbb{P}(\mathrm{First}) \cdot \mathbb{P}(\mathrm{Red} \mid \mathrm{First}) + \mathbb{P}(\mathrm{Second}) \cdot \mathbb{P}(\mathrm{Red} \mid \mathrm{Second}) + \mathbb{P}(\mathrm{Third}) \cdot \mathbb{P}(\mathrm{Red} \mid \mathrm{Third}) \\ &= \frac{1}{6} \cdot 1 + \frac{2}{6} \cdot \frac{1}{2} + \frac{3}{6} \cdot 0 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}\end{aligned}$$

$$\begin{aligned}\mathbb{P}(\mathrm{Red}) &= \mathbb{P}(\mathrm{First}) \cdot \mathbb{P}(\mathrm{Red} \mid \mathrm{First}) + \mathbb{P}(\mathrm{Second}) \cdot \mathbb{P}(\mathrm{Red} \mid \mathrm{Second}) + \mathbb{P}(\mathrm{Third}) \cdot \mathbb{P}(\mathrm{Red} \mid \mathrm{Third}) \\ &= \frac{1}{6} \cdot 0 + \frac{2}{6} \cdot \frac{1}{2} + \frac{3}{6} \cdot 1 = \frac{1}{6} + \frac{3}{6} = \frac{2}{3}\end{aligned}$$

b) If you see a red side, what is the probability that the other side of the card is blue?

This happens precisely when Jordan chose the second card. Therefore this probability is precisely $\mathbb{P}(\mathrm{Second} \mid \mathrm{Red})$.

By Bayes Theorem,

$$\begin{aligned}\mathbb{P}(\mathrm{Second} \mid \mathrm{Red}) &= \frac{\mathbb{P}(\mathrm{Red} \mid \mathrm{Second}) \cdot \mathbb{P}(\mathrm{Second})}{\mathbb{P}(\mathrm{Red})} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{3}} = \frac{1}{2}\,. \\ &= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{2}{3}} = \frac{1}{4}\,.\end{aligned}$$

Task 5 – Luggage swap **[25 pts]**

Alex, Barbara, and Carol have suitcases that look identical to each other but different from the suitcases of all the other passengers on their flight. At baggage claim they couldn't tell each other's suitcases apart and, one after another, each chose a uniformly random suitcase from the ones that were left that looked like theirs, so the suitcases were chosen in a uniformly random order.
Let X be the number of people from $\{\text{Alex, Barbara, Carol}\}$ who ended up with the wrong suitcase.

a) Give the PMF of X. (In particular, give the range Ω_X of X.)

We have

$$\Omega_X = \{0, 1, 2, 3\}.$$

Moreover, there are $3! = 6$ ways of choosing the suitcases, of which 2 leave everyone with wrong suitcases, 3 leave one person with the correct suitcase and two people with wrong suitcases, and 1 where everyone has the correct suitcase. (Note that it is not possible to have two people with the correct suitcases and one with the wrong suitcase). Therefore

$$p_X(0) = \frac{1}{6}$$
$$p_X(1) = 0$$
$$p_X(2) = \frac{3}{6} = \frac{1}{2}$$
$$p_X(3) = \frac{2}{6} = \frac{1}{3}$$

b) Compute $\mathbb{E}[X]$ directly using **a)**.

By definition

$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2) + 3 \cdot p_X(3) = 2 \cdot 1/2 + 3 \cdot 1/3 = 2$$

c) Compute $\mathbb{E}[X]$ using the linearity of expectation and indicator random variables.

Define X_A, X_B, and X_C to be the indicator r.v.'s for the events that Alex, Barbara, and Carol, respectively, do *not* get the correct suitcase. Then, by linearity

$$\mathbb{E}[X] = \mathbb{E}[X_A] + \mathbb{E}[X_B] + \mathbb{E}[X_C]$$

For each of Alex, Barbara, and Carol, out of the 6 ways of choosing suitcases, exactly four leave them with the wrong suitcase. Thus, $\mathbb{E}[X_A] = \mathbb{E}[X_B] = \mathbb{E}[X_C] = \frac{4}{6} = \frac{2}{3}$, and $\mathbb{E}[X] = 3 \times 2/3 = 2$.

d) What is $\mathbb{E}\left[X^2\right]$? Simplify the result as far as possible.

By definition

$$\begin{aligned}\mathbb{E}[X^2] &= 0 \cdot p_X(0) + 1^2 \cdot p_X(1) + 2^2 \cdot p_X(2) + 3^3 \cdot p_X(3) \\ &= 4 \cdot \frac{1}{2} + 9 \cdot \frac{1}{3} \\ &= 5\end{aligned}$$

e) What is $\text{Var}(X)$? (Try to compute this as efficiently as possible.)

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 5 - 2^2 = 1$$

Task 6 – Safety Cards [25 pts]

Alex, Barbara, and Carol are on a game show. On the show, a dealer shuffles 3 cards labeled A, B, C and randomly and uniformly deals one card to each of Alex, Barbara, and Carol. If the player gets a card with a letter that begins their first name, they are safe, otherwise they are "at risk" and must answer a skill-testing question.
Let X be the number of contestants among Alex, Barbara, and Carol who are at risk.

a) Give the PMF of X. (In particular, give the range Ω_X of X.)

We have

$$\Omega_X = \{0, 1, 2, 3\}.$$

Moreover, there are $3! = 6$ deals, of which 2 leave everyone at risk, 3 deals leave one player safe and two at risk, and 1 has everyone safe. (Note that it is not possible to have two players be safe without having the third one safe, as well!). Therefore

$$p_X(0) = \frac{1}{6}$$
$$p_X(1) = 0$$
$$p_X(2) = \frac{3}{6} = \frac{1}{2}$$
$$p_X(3) = \frac{2}{6} = \frac{1}{3}$$

b) Compute $\mathbb{E}[X]$ directly using **a)**.

By definition

$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2) + 3 \cdot p_X(3) = 2 \cdot 1/2 + 3 \cdot 1/3 = 2$$

c) Compute $\mathbb{E}[X]$ using the linearity of expectation and indicator random variables.

Define X_A, X_B, and X_C to be the indicator r.v.'s for the events that Alex, Barbara, and Carol, respectively, do *not* receive a card with their initial on it. Then, by linearity

$$\mathbb{E}[X] = \mathbb{E}[X_A] + \mathbb{E}[X_B] + \mathbb{E}[X_C]$$

For each of Alex, Barbara, and Carol, out of 6 deals, exactly four do not give them a card with their own initial on it. Thus, $\mathbb{E}[X_A] = \mathbb{E}[X_B] = \mathbb{E}[X_C] = \frac{4}{6} = \frac{2}{3}$, and $\mathbb{E}[X] = 3 \times 2/3 = 2$.

d) What is $\mathbb{E}\left[X^2\right]$? Simplify the result as far as possible.

By definition

$$\begin{aligned}
\mathbb{E}[X^2] &= 0 \cdot p_X(0) + 1^2 \cdot p_X(1) + 2^2 \cdot p_X(2) + 3^3 \cdot p_X(3) \\
&= 4 \cdot \frac{1}{2} + 9 \cdot \frac{1}{3} \\
&= 5
\end{aligned}$$

e) What is $\text{Var}(X)$? (Try to compute this as efficiently as possible.)

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 5 - 2^2 = 1$$