Student name: ____________ **Student Number:** ____________

CSE 312: Foundations of Computing II - Autumn 2022

Midterm Exam - Practice

Solutions

Important: Do not read the following pages until instructed to start. Read the instructions on this page carefully.

Instructions. This midterm is meant to be solved in 50 minutes. **The start time is your lecture start time and you are required to stop writing at** your lecture end time, unless otherwise instructed. When done, wait for a proctor to collect your solution, or follow any other instructions by the proctors.

- This exam consists of **five tasks**, overall with **100 points**.
- **Write your name and student number on top of this page**
- This is a **closed-book exam.** No written document is allowed. In particular, you are not allowed to use any textbooks, nor additional personal notes, homework assignments or solutions, etc.
- A detachable cheat sheet is provided on the front and back of the last page of this midterm.
- **No electronics are allowed** during the exam (no smart phones, no laptops, no smart watches, no pocket calculators, etc). Keep them stored in your bag.
- Write your solutions in the appropriate spaces (additional empty space is available at the end of the midterm – please reference to it if you use this space).

Good luck!

Task 1 – Short Questions [20 pts]

Are the following statements True or False? Provide a <u>short</u> justification for your answer, explaining why the statement is true or false.

a) $\binom{1024}{4} = \binom{1023}{4} + \binom{1023}{3}$.

> **True**. The left side counts the number of ways to choose 4 numbers from 1 to 1024. This set of 4 numbers can either exclude the number 1024 in which case there are $\binom{1023}{4}$ possible sets or include the number 1024 in which case there are $\binom{1023}{3}$ possible way of filling out the remaining 3 elements.

b) Let X and Y be finite sets. Then, $|X \backslash Y| = |X| - |Y|$.

> **False**. Y may contain elements that aren't in X. For example $X = \{1\}$ and $Y = \{1, 2\}$. Then the right side would be negative, which is impossible.

c) There exist events $\mathcal{A}$ and $\mathcal{B}$ (defined on the same probability space $(\Omega, \mathbb{P})$) such that $\mathbb{P}(\mathcal{A}) = 0.8$, $\mathbb{P}(\mathcal{B}) = 0.5$, and $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = 0.2$.

> **False**. If this were possible, by inclusion-exclusion we would have
>
> $$\mathbb{P}(\mathcal{A} \cup \mathcal{B}) = \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{B}) - \mathbb{P}(\mathcal{A} \cap \mathcal{B}) = 0.8 + 0.5 - 0.2 = 1.1$$
>
> which is not a legal probability.

d) if A and B are events such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$ then A and B cannot be both independent and mutually exclusive.

True. If A and B are mutually exclusive then $\mathbb{P}(A \cap B) = 0$, but we are given then $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, so $\mathbb{P}(A) \cdot \mathbb{P}(B) > 0 = \mathbb{P}(A \cap B)$ so A and B cannot be independent.

e) If X and Y are random variables such that $\mathbb{E}[X+Y] = \mathbb{E}[X]+\mathbb{E}[Y]$ then X and Y are independent.

False. $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ is always true for every pair of random variables X and Y including ones that are not independent.

Task 2 – Counting [20 pts]

For both tasks, your final answers can contain factorials and/or binomial coefficients. However, do evaluate small factorials (up to $3! = 6$).

a) What is the number of re-arrangements of the word MIDTERM that contain no two consecutive vowels?

We can count this by figuring out where among the 7 letters the vowels I and E go and then, for each placement of I and E figuring out the remaining 5 positions. There are $\binom{7}{2} = \frac{7\times 6}{2} = 21$ pairs of positions in the final string but 6 of them $(\{1,2\},\ldots,\{6,7\})$ have vowels next to each other, so there $21 - 6 = 15$ places for the vowels. For each placement for vowels there are 2 orders for the I and E. For the 5 other letters, the letter M appears twice and all the other letters are the different so there are $5!/2!$ re-arrangements for the other letters. This gives a total of

$$15 \cdot 2 \cdot 5!/2! = 15 \cdot 5!$$

re-arrangements.

b) From a group of 20 people, we need to choose two committees – Committee A, of 5 people, and Committee B, of 7 people – which can share at most *one* member.

How many ways are there of doing so?

There are $\binom{20}{5}$ ways to choose committee A. For each such choice: If there is no overlap we have $\binom{15}{7}$ ways to choose Committee B from the 15 remaining members. A Committee B with overlap of one member would have 5 choices of the overlapping member and $\binom{15}{6}$ ways of choosing the remaining 6 members. By the product and sum rules we have a total of

$$\binom{20}{5} \cdot \left(\binom{15}{7} + 5 \cdot \binom{15}{6}\right)$$

committees.

Task 3 – Basic Probability [15 pts]

A jar contains 3 red, 5 green, and 2 blue balls. Three times in a row, we pick a random ball (uniformly) from the jar, and then discard it (i.e., we do not add it back to the jar). Recall that this is referred to as sampling without replacement.

a) What is the probability that all balls are red?

The balls chosen are a uniformly random subset of 3 balls from the original 10 balls in the jar. There are $\binom{10}{3}$ subsets of 3 balls and each subset has a probability of $\frac{1}{\binom{10}{3}}$. Exactly 1 subset consists of 3 red balls, so the probability is

$$\frac{1}{\binom{10}{3}}$$

b) What is the probability that all three balls have the same color?

There is 1 subset with all red balls, $\binom{5}{3}$ subsets with all green balls and no subsets of all blue balls. Therefore the total probability is

$$\frac{1+\binom{5}{3}}{\binom{10}{3}}$$

c) What is the probability that both blue balls are chosen, along with a red ball?

The set must contain both of the two blue balls. To complete the set of balls chosen, there are 3 choices of red ball to add to the two blue balls so there are precisely 3 possible sets of three balls chosen to yield this event. Therefore the probability is

$$\frac{3}{\binom{10}{3}}$$

Task 4 – Conditional Probabilities **[20 pts]**

Alice has two six-sided dice. The first die has all numbers between 1 and 6 on its sides. The second die has been manipulated, and only the even numbers 2, 4, 6 appear on its sides, i.e., each of the three numbers appears on *two* sides.

Assume that Alice picks the first die with probability $2/3$, and the second die with probability $1/3$. She then rolls the selected die, and we can assume that the die roll itself is fair, i.e., all sides are equally likely. We are interested in the number obtained from the roll of the chosen die.

a) What is the probability that Alice obtains an even number?

Let Even be the event that Alice rolls an even number and First be the probability that Alice chooses the first die. We use the law of total probability.

$$\begin{aligned}\mathbb{P}(\mathrm{Even}) &= \mathbb{P}(\mathrm{Even} \mid \mathrm{First}) \cdot \mathbb{P}(\mathrm{First}) + \mathbb{P}(\mathrm{Even} \mid \mathrm{First}^c) \cdot \mathbb{P}(\mathrm{First}^c) \\ &= 1/2 \cdot 2/3 + 1 \cdot 1/3 = 2/3\end{aligned}$$

b) If Alice sees obtains an even number, what is the probability that the first die was rolled?

Using Bayes theorem, we have

$$\mathbb{P}\left(\mathrm{First} \,\middle|\, \mathrm{Even}\right) = \frac{\mathbb{P}\left(\mathrm{Even} \,\middle|\, \mathrm{First}\right) \cdot \mathbb{P}(\mathrm{First})}{\mathbb{P}(\mathrm{Even})} = \frac{2/3 \cdot 1/2}{2/3} = \frac{1}{2}$$

c) Are the events "an odd number is chosen" and "the second die is rolled" independent?

We have $\mathbb{P}(\mathrm{Odd}) = 1/3$, $\mathbb{P}(\mathrm{Second}) = 1/3$. But also $\mathbb{P}(\mathrm{Second} \cap \mathrm{Odd}) = 0$. Therefore, the two events are not independent.

Task 5 – Red Balls **[25 pts]**

An urn contains 3 white and 3 red balls. Eve picks 2 balls from this urn at random *without* replacement. Let R be the number of red balls among the two balls she picks.

a) Give the PMF of R. (In particular, give the range Ω_R of R.)

There are two balls chosen and we have enough balls that each of these can be red or white so

$$\Omega_R = \{0, 1, 2\}.$$

$$p_R(0) = \frac{\binom{3}{2}}{\binom{6}{2}} = \frac{3}{\binom{6}{2}} = \frac{3}{15} \quad \text{since 2 of the 3 white balls must be chosen}$$

$$p_R(1) = \frac{3^2}{\binom{6}{2}} = \frac{9}{\binom{6}{2}} = \frac{9}{15} \quad \text{since 1 of the 3 white balls and 1 of the 3 red balls must be chosen}$$

$$p_R(2) = \frac{\binom{3}{2}}{\binom{6}{2}} = \frac{3}{\binom{6}{2}} = \frac{3}{15} \quad \text{since 2 of the 3 red balls must be chosen}$$

b) Compute $\mathbb{E}[R]$ directly using **a)**.

By definition

$$\begin{aligned}
\mathbb{E}[R] &= 0 \cdot p_R(0) + 1 \cdot p_R(1) + 2 \cdot p_R(2) \\
&= 1 \cdot \frac{9}{15} + 2 \cdot \frac{3}{15} \\
&= \frac{15}{15} \\
&= 1
\end{aligned}$$

c) Compute $\mathbb{E}[R]$ using the linearity of expectation and indicator random variables.

Define R_1 to be the indicator r.v. for the first ball being red and R_2 for the indicator r.v. for the second ball being red. Then $R = R_1 + R_2$. By linearity of expectation we have

$$\mathbb{E}[R] = \mathbb{E}[R_1] + \mathbb{E}[R_2.]$$

Each ball chosen is equally likely to be any of the balls. Since there are an equal number of white and red balls, the probability that each is red is $1/2$. Therefore $\mathbb{E}[R_1] = \mathbb{E}[R_2] = 1/2$, so

$$\mathbb{E}[R] = \mathbb{E}[R_1] + \mathbb{E}[R_2] = 1/2 + 1/2 = 1$$

d) What is $\mathbb{E}[R^2]$? Simplify the result as far as possible.

By definition

$$\begin{aligned}\mathbb{E}[R] &= 0 \cdot p_R(0) + 1^2 \cdot p_R(1) + 2^2 \cdot p_R(2) \\ &= 1 \cdot \frac{9}{15} + 4 \cdot \frac{3}{15} \\ &= \frac{21}{15} \\ &= \frac{7}{5}\end{aligned}$$

e) What is $\mathrm{Var}(R)$? (Try to compute this as efficiently as possible.)

$$\mathrm{Var}(R) = \mathbb{E}[R^2] - \mathbb{E}[R]^2 = \frac{7}{5} - 1^2 = \frac{2}{5}$$