Student name: ______________ **Student Number:** ______________

CSE 312: Foundations of Computing II - Autumn 2022

Midterm Exam

Solutions

Important: Do not read the following pages until instructed to start. Read the instructions on this page carefully.

Instructions. This midterm is meant to be solved in 50 minutes. **The start time is 1:30pm** and you are required to stop writing at **2:20pm,** unless otherwise instructed. When done, wait for a proctor to collect your solution, or follow any other instructions by the proctors.

- This exam consists of **five tasks**, overall with **100 points**.
- **Write your name and student number on top of this page**
- This is a **closed-book exam.** No written document is allowed. In particular, you are not allowed to use any textbooks, nor additional personal notes, homework assignments or solutions, etc.
- A detachable cheat sheet is provided on the front and back of the last page of this midterm. Please be gentle when removing it.
- **No electronics are allowed** during the exam (no smart phones, no laptops, no smart watches, no pocket calculators, etc). Keep them stored in your bag.
- Write your solutions in the appropriate spaces (additional empty space is available at the end of the midterm – **please add a pointer to where the rest of your solution is if you used any extra space**).

Good luck!

Task 1 – Short Questions [20 pts]

Are the following statements True or False? Provide a short justification for your answer, explaining why the statement is true or false.

a) $\sum_{k=0}^{n} \binom{n}{k} 4^k = 5^n$.

True. The binomial theorem says that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. We can obtain the identity by setting $x = 4$ and $y = 1$.

b) If A, B, and C are events then $\mathbb{P}(A \cup B \cup C) \leqslant \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$.

True. Solution 1:

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c) + \mathbb{P}(C \cap A^c \cap B^c) \leqslant \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \; .$$

Solution 2: By inclusion-exclusion $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leqslant \mathbb{P}(A) + \mathbb{P}(B)$ since $\mathbb{P}(A \cap B) \geqslant 0$. We just apply this twice to get

$$\mathbb{P}(A \cup B \cup C) \leqslant \mathbb{P}(A \cup B) + \mathbb{P}(C) \leqslant \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) .$$

Solution 3: $\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C)$ counts the probability of every outcome in $A \cup B \cup C$ at least once. (It is potentially larger because it counts outcomes that occur in exactly two of A, B, C twice and it counts outcomes that occur in all three of A, B, C three times.) .

c) Let A and B be events such that $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.6$ and $\mathbb{P}(A \cup B) = 0.9$. Are the events A and B independent?

False. We can use inclusion-exclusion to compute

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.5 + 0.6 - 0.9 = 0.2 \; .$$

At the same time, note that $\mathbb{P}(A) \cdot \mathbb{P}(B) = 0.5 \times 0.6 = 0.3 \neq 0.2$, and thus A, B are not independent.

d) Even without knowing any actual birthdays, we can be certain that in our CSE 312 class, which has more than 150 students registered, there must be at least 22 who were born on the same day of the week.

True. There are $k = 7$ possible days of the week for birthdays and $n \geqslant 150$ students. By the pigeonhole principle, there must one day of the week on which at least $\lceil \frac{150}{7} \rceil = 22$ students were born.

e) If X and Y are independent random variables, each taking values in $\{+1, -1\}$, then

$$\mathbb{E}\left[(X+Y)^2\right] = 2 + 2\,\mathbb{E}\left[X\right] \cdot \mathbb{E}\left[Y\right].$$

True. First, note that by linearity of expectation

$$\mathbb{E}\left[(X+Y)^2\right] = \mathbb{E}\left[X^2 + 2XY + Y^2\right] = \mathbb{E}\left[X^2\right] + 2\,\mathbb{E}\left[XY\right] + \mathbb{E}\left[Y^2\right] \; .$$

Because X, Y are independent, we also have $\mathbb{E}\left[XY\right] = \mathbb{E}\left[X\right] \cdot \mathbb{E}\left[Y\right]$. Because each of X and Y is 1 or -1, $X^2 = Y^2 = 1$.

Task 2 – Counting [15 pts]

For both tasks, your final answers can contain factorials and/or binomial coefficients. However, do evaluate small factorials (up to $3! = 6$).

a) A CSE 312 baseball team consists of 9 distinct positions.
How many ways are there to designate these distinct positions as either for a P (professor), T (teaching assistant), or S (student) if we require that the team consist of

- 1 professor,
- 3 teaching assistants (TAs), and
- 5 students?

This is the same as counting the number of distinct re-orderings of the string PTTTSSSSS which is $\binom{9}{1,3,5} = \frac{9!}{3!\ 5!}$.
Alternatively, one could count this by taking the 9 choices for the position designated for the professor and then, for each of those choices, choosing a subset of 3 of the remaining 8 for the TAs. By the product rule this gives $9 \cdot \binom{8}{3}$, which is the same value.

b) Xinyue is ordering 20 lunches for a group lunch. There are 5 options for each lunch. Of these options, 3 are sandwiches, Chicken, Kafta, or Falafel, and 2 are salads, Greek or Tabbouleh. A lunch order consists of the numbers of each option it includes.
How many different lunch orders does Xinyue have to choose from?

There are 5 options for each lunch and the only thing that matters is how many of each lunch option is chosen. This is a sequence of 5 non-negative integers whose sum is 20, so there are $\binom{20+5-1}{5-1} = \binom{24}{4}$ options. (The fact that some options are sandwiches and some are salads doesn't matter.)

Task 3 – Basic Probability [20 pts]

A jar contains four red balls numbered 1 to 4, four blue balls numbered 1 to 4, and four green balls numbered 1 to 4. We choose three balls from the jar, one after another, uniformly at random, without replacing any after each choice.

a) What is the probability that the balls we chose were a red 1, a blue 2, and a green 3 in that order?

There is a $\frac{1}{12}$ probability of getting a red 1 on the first draw. Given that, there is a $\frac{1}{11}$ for that to be followed by a blue 2, and a $\frac{1}{10}$ probability after that that there third ball would be a green 3. Therefore the total probability is $\frac{1}{12 \cdot 11 \cdot 10}$.

Alternatively: the balls drawn in order form a uniformly random 3-permutation of a set of size 12. This event corresponds to precisely one such 3-permutation, so the probability is 1 over the number of such 3-permutations, which is $1/(\frac{12!}{(12-3)!}) = \frac{9!}{12!} = \frac{1}{12 \cdot 11 \cdot 10}$.

b) What is the probability that the three balls we chose include a red ball and a blue ball that have the same number?

If we have red and blue balls with the same number then they could be in any one of the 3 pairs of positions. (Once there is one such pair, there cannot be another so these are disjoint events.) There are 4 possible red-blue pairs out of the $\binom{12}{2}$ pairs in those positions. The total probability is 3 times the probability for each pair having the same number or

$$\frac{3 \cdot 4}{\binom{12}{2}}.$$

c) What is the probability that not all balls are the same color?

This probability is 1 minus the probability that the balls are all the same color. Among the $\binom{12}{3}$ subsets of 3 balls that may be chosen, for each color there are $\binom{4}{3}$ ways to choose 3 balls of that color and these cases are disjoint. Therefore this probability is

$$1 - \frac{3 \cdot \binom{4}{3}}{\binom{12}{3}}$$

Task 4 – Conditional Probabilities **[20 pts]**

Percy's restaurant offers a chance at different discounts on dinner depending on the day of the week. At the end of the meal, each customer gets a random chance at either a Gold or Silver discount. (They may get no discount at all.)
There are three cases depending on the day of the week:

A. On Sundays there is a 1/2 chance of getting a Silver discount and a 1/2 chance of getting a Gold discount.

B. On Mondays or Tuesdays there is a 2/3 chance of getting a Silver discount and a 1/4 chance of getting a Gold discount.

C. On Wednesdays, Thursdays, Fridays, or Saturdays there is a 1/3 chance of getting a Silver discount and a 1/4 chance of getting a Gold discount.

Suppose that Tanmay chose one of the 7 days of the week to go to Percy's restaurant uniformly at random:

a) What is the probability that Tanmay got a Gold discount?

Let Gold be the event that got a Gold discount. Let events A, B, and C be the events that you chose the corresponding cases. By the LTP, we have

$$\begin{aligned}\mathbb{P}(\mathrm{Gold}) &= \mathbb{P}(\mathrm{A}) \cdot \mathbb{P}(\mathrm{Gold} \mid \mathrm{A}) + \mathbb{P}(\mathrm{B}) \cdot \mathbb{P}(\mathrm{Gold} \mid \mathrm{B}) + \mathbb{P}(\mathrm{C}) \cdot \mathbb{P}(\mathrm{Gold} \mid \mathrm{C}) \\ &= \frac{1}{7} \cdot \frac{1}{2} + \frac{2}{7} \cdot \frac{1}{4} + \frac{4}{7} \cdot \frac{1}{4} = \frac{1}{14} + \frac{1}{14} + \frac{1}{7} = \frac{2}{7}\end{aligned}$$

b) If all you know is that Tanmay ended up with a Gold discount, what is the probability that Tanmay went to Percy's on a Sunday?

This is precisely $\mathbb{P}(\mathrm{A} \mid \mathrm{Gold})$. By Bayes Theorem,

$$\begin{aligned}\mathbb{P}(\mathrm{A} \mid \mathrm{Gold}) &= \frac{\mathbb{P}(\mathrm{Gold} \mid \mathrm{A}) \cdot \mathbb{P}(\mathrm{A})}{\mathbb{P}(\mathrm{Gold})} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{7}}{\frac{2}{7}} = \frac{1}{4}\,.\end{aligned}$$

Task 5 – The Check Isn't in the Mail [25 pts]

Paul is a bit old-fashioned and sometimes pays his bills by sending checks in the mail. He carefully writes out and signs each check. He did this for 3 bills this month: Tax, Water, and Gas. However, he was very distracted by the need to compose the CSE 312 midterm and didn't pay attention to which envelopes he was putting them in before sealing them up and putting them in the mail. They ended up scrambled, uniformly at random, one per envelope. Let X be the number of Paul's bills that got sent with the wrong check in them.

a) Give the PMF of X. (In particular, give the range Ω_X of X.)

We have

$$\Omega_X = \{0, 2, 3\}.$$

There are $3! = 6$ ways of putting the checks in the envelopes, of which there is 1 with all the envelopes having the correct check in them, 3 having one envelope with the correct check and two envelopes with the wrong check, and 2 having every check in the wrong envelope. (Note that it is not possible to have two checks in the correct envelope and one in the wrong envelope.) Therefore

$$p_X(0) = \frac{1}{6}$$
$$p_X(2) = \frac{3}{6} = \frac{1}{2}$$
$$p_X(3) = \frac{2}{6} = \frac{1}{3}$$

b) Compute $\mathbb{E}[X]$ directly using **a)**.

By definition

$$\begin{aligned}\mathbb{E}[X] &= 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2) + 3 \cdot p_X(3) \\ &= 2 \cdot 1/2 + 3 \cdot 1/3 = 2.\end{aligned}$$

c) Compute $\mathbb{E}[X]$ using the linearity of expectation and indicator random variables.

Define X_T, X_W, and X_G to be the indicator r.v.'s for the events that the Tax, Water, and Gas envelopes , respectively, contain the *wrong check. Then, by linearity*

$$\mathbb{E}[X] = \mathbb{E}[X_T] + \mathbb{E}[X_W] + \mathbb{E}[X_G]$$

For each of the bills, Tax, Water, and Gas, out of the 6 ways of filling the envelopes with checks, exactly 4 leave them with the wrong check because there are 2 choices for the incorrect check in the envelope and then 2 further choices for how the other two checks ended up. Thus, $\mathbb{E}[X_T] = \mathbb{E}[X_W] = \mathbb{E}[X_G] = \frac{4}{6} = \frac{2}{3}$*, and* $\mathbb{E}[X] = 3 \times 2/3 = 2$*.*

d) What is $\mathbb{E}\left[X^2\right]$? Simplify the result as far as possible.

By definition

$$\begin{aligned}\mathbb{E}[X^2] &= 0 \cdot p_X(0) + 2^2 \cdot p_X(2) + 3^2 \cdot p_X(3) \\ &= 4 \cdot \frac{1}{2} + 9 \cdot \frac{1}{3} \\ &= 5\end{aligned}$$

e) What is $\text{Var}(X)$? (Try to compute this as efficiently as possible.)

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 5 - 2^2 = 1$$