Student name: ________________ **Student Number:** ________________

CSE 312: Foundations of Computing II - Winter 2024

Midterm A

Solutions

Important: Do not turn the page until instructed to start. In the meantime, read the instructions on this page carefully.

Instructions. You have 50 minutes to complete this midterm.

- **Write your name and student number on top of this page.**
- This is a **closed-book exam** with the exception of a one page cheat sheet that you may bring.
- **No electronics are allowed** during the exam (no smart phones, no laptops, no smart watches, no pocket calculators, etc). Before you come into the room where you are taking the test, store them in your bag/backpack and do not take them out until you leave the room after the exam. If we see any such items, we will take your test away.
- Write your solutions in the appropriate spaces (additional empty space is available at the end of the midterm – **please add a pointer to where the rest of your solution is if you used any extra space**).
- If your answer to a question is correct, you will get full credit *regardless of whether or not you provide any explanation*. If your final answer is wrong though, you may get partial credit if you provide correct partial explanations.
- If a problem looks difficult, I recommend moving on to another problem and coming back later.

Good luck!

Task 1 – Short answer **[48 pts]**

(8 points each)

When answering the following questions, you do NOT need to explain your answers, but please **put a box around your final answer.**

a) Let A and B be events such that $\mathbb{P}(A) = 0.5$, $\mathbb{P}(B) = 0.6$ and $\mathbb{P}(A \cup B) = 0.8$. What is $\mathbb{P}(A \cap B)$?

We can use inclusion-exclusion to compute:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.5 + 0.6 - 0.8 = 0.3\,.$$

b) Suppose that X is a random variable s.t. $\Omega_X = \{-2, +2\}$. Fill in the following probabilities so that $\mathbb{E}[X] = 1$.

$$p_X(x) = \begin{cases} \ldots\ldots & \text{if } x = 2 \\ \ldots\ldots & \text{if } x = -2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $p = p_X(2)$. Then

$$\mathbb{E}[X] = 2p - 2(1-p) = 1.$$

So

$$p = 3/4.$$

In other words,

$$p_X(x) = \begin{cases} 0.75 & \text{if } x = 2 \\ 0.25 & \text{if } x = -2 \\ 0 & \text{otherwise.} \end{cases}$$

c) Suppose that Y is a random variable with $\Omega_Y = \{-2, +2\}$. Circle one of True or False for **each** of the following three statements. More than one of them might be True.

True or False: $\mathbb{E}\left[Y^2\right] = 2$.

False

True or False: $\mathbb{E}\left[Y^2\right] = 4$.

True. Since $\Omega_Y = \{-2, +2\}$, Y^2 is always equal to 4.

True or False: There is not enough information provided to compute Var(Y).

True. We need to know either the pmf or the expectation to compute the variance with the given information.

d) Suppose that X and Y are independent random variables such that $\Omega_X = \{-2, +2\}$ and $\Omega_Y = \{-2, +2\}$, where $\mathbb{E}[X] = 1$ and $\mathbb{E}[Y] = -1$. Compute $\mathbb{E}\left[10 \cdot (X-Y)^2\right]$. Your answer should be a number. (Hint: Expand the expression inside the expectation and use properties of expectation.)

By linearity of expectation

$$\mathbb{E}\left[10 \cdot (X-Y)^2\right] = 10 \cdot \mathbb{E}\left[X^2 - 2XY + Y^2\right] = 10(\mathbb{E}\left[X^2\right] - 2\,\mathbb{E}[XY] + \mathbb{E}\left[Y^2\right]) .$$

Because X, Y are independent, we also have $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$. Because each of X and Y is 2 or -2, $X^2 = Y^2 = 4$. Therefore

$$\mathbb{E}\left[10(X-Y)^2\right] = 10(8 - 2\,\mathbb{E}[X] \cdot \mathbb{E}[Y]) = 100.$$

e) Consider a class with n students. Their homeworks are handed back according to a uniformly random permutation. Then

$$\mathbb{P}(\text{exactly } k \text{ people get their own homework back}) = \frac{\binom{n}{k}(n-k)!}{n!} \tag{1}$$

Circle the correct statement among (i), (ii) or (iii).

(i) This is an overestimate.

(ii) This is an underestimate.

(iii) This is correct.

The correct solution is (i). The denominator correctly counts the size of the sample space. But the numerator counts cases where more than k people get their own homework back. For example, it counts the case that everybody gets their own homework back $\binom{n}{k}$ times!

Let's see how to compute the correct answer to this question.

Let's first compute N_m, the number of ways to return m homeworks to m people such that *nobody* gets their own homework back. Let's say A_i is the number of ways to return homeworks so that person i gets their homework back. Then by inclusion-exclusion,

$$N_m = m! - \sum_i A_i + \sum_{i,j} A_i \cap A_j - \sum_{i,j,k} A_i \cap A_j \cap A_k + \dots$$

Now $A_i = (m-1)!$, $A_i \cap A_j = (m-2)!$, $A_i \cap A_j \cap A_k = (m-3)!$ and so on. Therefore,

$$N_m = m! - \binom{m}{1}(m-1)! + \binom{m}{2}(m-2)! - \binom{m}{3}(m-3)! + \dots = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!}.$$

Coming back to our original problem, we want to compute the probability that exactly k people get their own homework back. This is

$$\frac{\binom{n}{k} N_{n-k}}{n!} = \frac{\binom{n}{k}(n-k)! \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}}{n!} = \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}$$

f) How many ways are there to divide a team of 10 people into 3 groups, one of size 2 and two of size 4?

Note that the people are distinguishable from one another, but order of those people doesn't matter within the groups and order doesn't matter between the groups. Thus, for example, if the people on the team are named $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, then the following three ways of dividing up the team are all the same.

(1,2) (3,4,5,6) (7,8,9,10)

(1,2) (7,8,9,10) (3,4,5,6)

(3,4,5,6) (8,7,9,10) (1,2)

but the above are different from

(3,4,5,8) (6,7,9,10) (1,2)

$$\frac{\binom{10}{4}\binom{6}{4}}{2}.$$

We first choose a group of size 4, call it A, then choose another group of size 4, call it B. The 2 leftover will be the group of size 2. But every outcome will be counted twice since A,B could be selected in either order.

There are many other correct answers. For example:

$$\frac{\binom{10}{2}\binom{8}{4}}{2}.$$

$$\frac{\binom{10}{4}\binom{6}{2}}{2}.$$

$$\frac{\binom{10}{4,4,2}}{2}.$$

Task 2 – Committees [20 pts]

Consider a group of 100 people consisting of 10 left-handed people and 90 right-handed people. Suppose that we form a committee in the following manner: With probability 0.3, we select a subset of exactly 5 people out of the 100 people, uniformly at random. Otherwise, with probability 0.7, we select a subset of exactly 7 people out of the 100, again uniformly at random.
Now consider the following events:

- Let L be the event that the committee consists entirely of left-handed people
- Let F be the event that we selected exactly 5 people to be on the committee

When answering the following questions, you do NOT need to explain your answers and you do NOT need to simplify your answers, but please **put a box around your final answer.**

a) (10 points) What is the probability that the entire committee is left-handed, i.e. what is $\mathbb{P}(L)$? **Do not simplify your answer.**

Using the law of total probability, we have

$$P(F) \cdot P(L \mid F) + P(\overline{F}) \cdot P(L \mid \overline{F}) = 0.3 \cdot \frac{\binom{10}{5}}{\binom{100}{5}} + 0.7 \cdot \frac{\binom{10}{7}}{\binom{100}{7}}$$

b) (10 points) Let q be the correct answer to part a). What is $\mathbb{P}(F \mid L)$, that is, the probability that we selected a committee of size 5, given that the entire committee is left-handed? Write your answer in terms of q.

$$P(F|L) = \frac{P(L \mid F)P(F)}{P(L)} = \frac{\frac{\binom{10}{5}}{\binom{100}{5}} \cdot 0.3}{q}$$

Task 3 – Sharing birthdays [32 pts]

Suppose that the birthdays of 200 people are each independently and equally likely to be any of 365 possible days of the year. Answer the following six questions about this scenario. You do NOT need to explain your answers and you do NOT need to simplify your answers, but please **put a box around each of your final answers.**

a) (4 points) Let X_{Jan1} be the number of people whose birthday is January 1. What kind of random variable from our zoo is X_{Jan1} and what are its parameters?

$X_{\text{Jan1}} \sim \text{Bin}(200, \frac{1}{365})$

b) (4 points) What is $\mathbb{P}(X_{\text{Jan1}} = 5)$, that is, what is the probability that there are exactly 5 people whose birthday is Jan 1?

$\binom{200}{5} \left(\frac{1}{365}\right)^5 \left(1 - \frac{1}{365}\right)^{195}$

c) (5 points) Let X_{Feb1} be the number of people whose birthday is February 1. Are X_{Jan1} and X_{Feb1} independent random variables? Circle one of

Yes or **No**

No. For example if $X_{\text{Jan1}} = 200$, then $X_{\text{Feb1}} = 0$.

d) (4 points) Let X count the number of birthdays, out of the 365 possibilities, that are shared by exactly 5 people. (A particular birthday, say Jan 1, is "shared" by exactly 5 people, if there are exactly 5 out of the 200 people whose birthday is Jan 1.)

True or False: X is a Binomial random variable.

False.

$$X = X_{\text{Jan1}} + X_{\text{Jan2}} + \ldots + X_{\text{Dec31}}$$

These indicator random variables are not independent.

e) (6 points) What is $\mathbb{E}[X]$?

Using linearity of expectation,

$$E(X_{\text{Jan1}}) + E(X_{\text{Jan2}}) + \ldots + E(X_{\text{Dec31}}) = 365 \cdot \binom{200}{5}\left(\frac{1}{365}\right)^5\left(1-\frac{1}{365}\right)^{195}$$

f) (9 points) What is $\mathbb{P}(X_{\text{Jan1}} = 5 \ \cap \ X_{\text{Feb1}} = 3)$?

That is, what is the probability that there are exactly 5 people whose birthday is January 1 **and** exactly 3 people whose birthday is February 1?

$$\mathbb{P}(X_{\text{Jan1}} = 5 \ \cap \ X_{\text{Feb1}} = 3) = \binom{200}{5}\binom{195}{3}\left(\frac{1}{365}\right)^8\left(1-\frac{2}{365}\right)^{192}$$

Alternatively, use conditional probability:

$$\mathbb{P}(X_{\text{Jan1}} = 5) \cdot Pr(X_{\text{Feb1}} = 3 | X_{\text{Jan1}} = 5) = \binom{200}{5}\left(\frac{1}{365}\right)^5\left(1-\frac{1}{365}\right)^{195} \cdot \binom{195}{3}\left(\frac{1}{364}\right)^3\left(1-\frac{1}{364}\right)^{192}$$

There are also other correct ways to write the above probability.