Section 8: Solutions

Review of Main Concepts

- **Multivariate: Discrete to Continuous:**

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, dsdt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dxdy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dxdy$
Independence must have	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

- **Law of Total Probability (r.v. version):** If X is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A | X = x) p_X(x) \qquad \text{discrete } X$$

- **Law of Total Expectation (Event Version)**: Let X be a discrete random variable, and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X | A_i] \mathbb{P}(A_i)$$

- **Conditional Expectation**: See table. Note that linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y | A] = \mathbb{E}[X | A] + \mathbb{E}[Y | A]$

- **Law of Total Expectation (RV Version)**: Suppose X and Y are random variables. Then,

$$\mathbb{E}[X] = \sum_{y} \mathbb{E}[X | Y = y] p_Y(y) \qquad \text{discrete version.}$$

- **Conditional distributions**

	Discrete	**Continuous**
Conditional PMF/PDF	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X\|Y=y] = \sum_x x p_{X\|Y}(x\|y)$	$\mathbb{E}[X\|Y=y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$

- **Continuous Law of Total Probability:**

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A | X = x) f_X(x) dx$$

- **Continuous Law of Total Expectation:**

$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X | Y = y] f_Y(y) dy$$

- **Markov's Inequality**: Let X be a non-negative random variable, and $\alpha > 0$. Then,

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

- **Chebyshev's Inequality**: Suppose Y is a random variable with $\mathbb{E}[Y] = \mu$ and $\text{Var}(Y) = \sigma^2$. Then, for any $\alpha > 0$,

$$\mathbb{P}(|Y - \mu| \geq \alpha) \leq \frac{\sigma^2}{\alpha^2}$$

- **(Multiplicative) Chernoff Bound**: Let $X_1, X_2, ..., X_n$ be *independent* Bernoulli random variables.

  Let $X = \Sigma_{i=1}^n X_i$, and $\mu = \mathbb{E}[X]$. Then, for any $0 \leq \delta \leq 1$,

  - $\mathbb{P}(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}$
  - $\mathbb{P}(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$

1. Content Review

(a) True or false: the Union Bound always gives a result in $[0, 1]$.
Solution:

False. Consider X and Y, which are independent indicator random variables.

Suppose $p_X(x) = \begin{cases} 0.75 & x = 0 \\ 0.25 & x = 1 \end{cases}$ and $p_Y(y) = \begin{cases} 0.75 & y = 0 \\ 0.25 & y = 1 \end{cases}$.

Then we may apply the Union Bound to place a bound on $P(X = 0 \cup Y = 0)$:

$$P(X = 0 \cup Y = 0) \leq P(X = 0) + P(Y = 0) = 0.75 + 0.75 = 1.5.$$

In these cases, the Union Bound tells us very little, since the probability of any event occurring is at most 1.

(b) True or false: Markov's Inequality always gives a non-negative result.
Solution:

True. Markov's Inequality is

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

as long as X is a non-negative random variable and $\alpha > 0$. Since X is a non-negative random variable, $\mathbb{E}[X] \geq 0$, so $\frac{\mathbb{E}[X]}{\alpha} \geq 0$.

(c) Suppose C and D are discrete random variables. Then $\mathbb{E}[C|D = d] =$

- ◯ $\sum_d d p_{D|C}(d|c)$
- ◯ $\sum_c c p_{D|C}(d|c)$
- ◯ $\int_{-\infty}^{\infty} c f_{c|d} dx$
- ◯ $\frac{\mathbb{E}[C]}{\mathbb{E}[D]}$

Solution:

Choice b is the correct answer from the definition of conditional expectation for discrete random variables.

(d) Suppose X and Y are random variables and A is an event. Given that $\mathbb{E}[X|A] = 4$ and $\mathbb{E}[Y|A] = 10$, what is $\mathbb{E}[2X + Y/2|A]$?

- ◯ 14
- ◯ 18

◯ 9

◯ 13

Solution:

Choice d is the correct answer since linearity of expectation still applies to conditional expectation:

$$\mathbb{E}\left[2X + Y/2|A\right] = \mathbb{E}\left[2X|A\right] + \mathbb{E}\left[Y/2|A\right] = 2\mathbb{E}\left[X|A\right] + \mathbb{E}\left[Y|A\right]/2 = 2 \cdot 4 + 10/2 = 8 + 5 = 13.$$

(e) True or false: Chebyshev's Inequality can best be described as giving an upper bound on the distribution's right tail.
Solution:

False. Chebyshev's Inequality gives an upper bound on the sum of the probabilties of the left and right tails of the distribution.

2. Tail bounds

Suppose $X \sim \text{Binomial}(6, 0.4)$. We will bound $\mathbb{P}(X \geq 4)$ using the tail bounds we've learned, and compare this to the true result.

(a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?
Solution:

We know that the expected value of a binomial distribution is np, so: $\mathbb{P}(X \geq 4) \leq \frac{\mathbb{E}[X]}{4} = \frac{2.4}{4} = 0.6$. We can use it since X is nonnegative.

(b) Give an upper bound for this probability using Chebyshev's inequality. You may have to rearrange algebraically and it may result in a weaker bound. **Solution:**

$\mathbb{P}(X \geq 4) = \mathbb{P}(X - 2.4 \geq 1.6) \leq \mathbb{P}(|X - 2.4| \geq 1.6)$ we can add those absolute value signs because that only adds more possible values, so it is an upper bound on the probability of $X - 2.4 \geq 1.6$. Then, using Chebyshev's inequality we get:
$\mathbb{P}(|X - 2.4| \geq 1.6) \leq \frac{Var(X)}{1.6^2} = \frac{1.44}{1.6^2} = 0.5625$

(c) Give an upper bound for this probability using the Chernoff bound. **Solution:**

First, we solve for the values of δ that will allow us to use the Chernoff bound. We want $(1+\delta)E[X] = (1+\delta)2.4 = 4$. Solving for δ here gives use $\delta = \frac{2}{3}$. Now, we can directly plug into the Chernoff bound. $\mathbb{P}(X \geq 4) = \mathbb{P}(X \geq (1+\frac{2}{3})2.4) \leq e^{-(\frac{2}{3})^2\mathbb{E}[X]/3} = e^{-4\times 2.4/27} \approx 0.7$

(d) Give the exact probability. **Solution:**

Since X is a binomial, we know it has a range from 0 to n (or in this case 0 to 6). Thus, the possible values to satisfy $X \geq 4$ are 4, 5, or 6. We plug in the PMF for each to get: $\mathbb{P}(X \geq 4) = \mathbb{P}(X = 4) + \mathbb{P}(X = 5) + \mathbb{P}(X = 6) = \binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6) + \binom{6}{6}0.4^6 \approx 0.1792$

3. Exponential Tail Bounds

Let $X \sim \text{Exp}(\lambda)$ and $k > 1/\lambda$.

(a) Use Markov's inequality to bound $\text{P}(X \geq k)$.

Solution:

We can use Markov's inequality here because X is non-negative since it is an exponential distribution. We also know that $E[X] = \lambda$ because $X \sim \text{Exp}(\lambda)$. By Markov's inequality, we get that:

$$\mathbb{P}(X \geq k) \leq \frac{1}{\lambda k}$$

(b) Use Markov's inequality to bound $\text{P}(X < k)$. **Solution:**

From Markov's inequality (and our answer in (a)), we know that $P(X \geq k) \leq \frac{1}{\lambda k}$. Then,

$$\begin{aligned} P(X \geq k) &\leq \frac{1}{\lambda k} \\ -P(X \geq k) &\geq -\frac{1}{\lambda k} && \text{multiplying be a negative flips the inequality} \\ 1 - P(X \geq k) &\geq 1 - \frac{1}{\lambda k} \\ P(X < k) &\geq 1 - \frac{1}{\lambda k} && \text{by definition of complement} \end{aligned}$$

Note that because we took the complement and the sign flipped, we have now found a *lower* bound for $P(X < k)$.

(c) Use Chebyshev's inequality to bound $\text{P}(X \geq k)$. **Solution:**

We rearrange algebraically to get into the form to apply Chebyshev's inequality. We then plug in the corresponding values and $Var(X) = \frac{1}{\lambda^2}$.

$$\mathbb{P}(X \geq k) = \mathbb{P}\left(X - \frac{1}{\lambda} \geq k - \frac{1}{\lambda}\right) \leq \mathbb{P}\left(\left|X - \frac{1}{\lambda}\right| \geq k - \frac{1}{\lambda}\right) \leq \frac{1}{\lambda^2(k - 1/\lambda)^2} = \frac{1}{(\lambda k - 1)^2}$$

(d) What is the exact formula for $\text{P}(X \geq k)$? **Solution:**

Using the CDF for an exponential distribution and definition of complement:

$$\mathbb{P}(X \geq k) = 1 - P(X \leq k) = 1 - (1 - e^{-\lambda k}) = e^{-\lambda k}$$

(e) For $\lambda k \geq 3$, how do the bounds given in parts (a), (c), and (d) compare?

Solution:

$$e^{-\lambda k} < \frac{1}{(\lambda k - 1)^2} < \frac{1}{\lambda k}$$

so Markov's inequality gives the worst bound.

4. Robbie's Late!

Suppose the probability Robbie is late to teaching lecture on a given day is at most 0.01. Do not make any independence assumptions.

(a) Use a Union Bound to bound the probability that Robbie is late at least once over a 30-lecture quarter. **Solution:**

Let R_i be the event Robbie is late to lecture on day i for $i = 1, ..., 30$. Then, by the union bound,

$$\begin{aligned}
\mathbb{P}(\text{late at least once}) &= \mathbb{P}(\bigcup_{i=1}^{30} R_i) \\
&\leq \sum_{i=1}^{30} \mathbb{P}(R_i) && [\text{union bound}] \\
&\leq \sum_{i=1}^{30} 0.01 && [\mathbb{P}(R_i) \leq 0.01] \\
&= 0.30
\end{aligned}$$

(b) Use a Union Bound to bound the probability that Robbie is **never** late over a 30-lecture quarter. **Solution:**

As in the previous part, let R_i be the event Robbie is late to lecture on day i for $i = 1, ..., 30$. Then, by the union bound, we found that

$$\mathbb{P}(\text{late at least once}) \leq 0.30$$

The probability Robbie is never late is the complement of the probability he is late at least once over the

30 lectures. Taking the complement and doing algebra:

$$\begin{aligned}
\mathbb{P}(\text{late at least once}) &\le 0.30 \\
-\mathbb{P}(\text{late at least once}) &\ge -0.30 && \text{[multiplying by negative flips the inequality]} \\
1 - \mathbb{P}(\text{late at least once}) &\ge 1 - 0.30 \\
\mathbb{P}(\text{never late}) &\ge 0.70
\end{aligned}$$

Note that we have now found a *lower* bound for this probability using the union bound because of taking the complement.

(c) Use a Union Bound to bound the probability that Robbie is late at least once over a 120-lecture quarter. **Solution:**

Let R_i be the event Robbie is late to lecture on day i for $i = 1, ..., 120$. Then, by the union bound,

$$\begin{aligned}
\mathbb{P}(\text{late at least once}) &= \mathbb{P}\Big(\bigcup_{i=1}^{120} R_i\Big) \\
&\le \sum_{i=1}^{120} \mathbb{P}(R_i) && \text{[union bound]} \\
&\le \sum_{i=1}^{120} 0.01 && [\mathbb{P}(R_i) \le 0.01] \\
&= 1.20
\end{aligned}$$

Notice that $\mathbb{P}(\text{late at least once}) \le 1.20$ is not a very helpful bound since probabilities have to be at most 1 already.

5. Trinomial Distribution

A generalization of the Binomial model is when there is a sequence of n independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let X_i be the number of times outcome i occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$. **Solution:**

We use a similar argument as for the binomial PMF. $\binom{n}{x_1,x_2,x_3}$ is the number of ways to select which of the n outcomes result in each of the 3 outcomes. Then, we multiply the probabilities of each trial being the corresponding outcome (e.g., $p_1^{x_1}$ is the probability that all x_1 trials end up being outcome 1). This gives use the following PMF:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} \prod_{i=1}^{3} p_i^{x_i} = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

where $x_1 + x_2 + x_3 = n$ and are nonnegative integers.

6. Do You “Urn” to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the i-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

(a) X_1, X_2 **Solution:**

Here is one way of defining the joint pmf of X_1, X_2

$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156}$$

$$\mathbb{P}(X_1 = 1, X_2 = 0) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 0 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{8}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 1) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 1 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{5}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 0) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{56}{156}$$

(b) X_1, X_2, X_3 **Solution:**

Instead of listing out all the individual probabilities, we could write a more compact formula for the pmf. In this problem, the denominator is always $P(13, k)$, where k is the number of random variables in the joint pmf. And the numerator is $P(5, i)$ times $P(8, j)$ where i and j are the number of 1s and 0s, respectively.

If we wish to compute $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$, then the number of 1s (i.e., white balls) is $x_1 + x_2 + x_3$, and the number of 0s (i.e., red balls) is $(1 - x_1) + (1 - x_2) + (1 - x_3)$. Then, we can write the pmf as follows:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{10!}{13!} \cdot \frac{5!}{(5 - x_1 - x_2 - x_3)!} \cdot \frac{8!}{(5 + x_1 + x_2 + x_3)!}$$

7. Successes

Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first 2 successes. Find the joint pmf of X_1 and X_2. Write an expression for $E[\sqrt{X_1 X_2}]$. You can leave your answer in the form of a sum. **Solution:**

X_1 and X_2 take on two particular values x_1 and x_2, when there are x_1 failures followed by one success, and then x_2 failures followed by one success. Since the Bernoulli trials are independent the joint pmf is

$$p_{X_1,X_2}(x_1, x_2) = (1 - p)^{x_1} p \cdot (1 - p)^{x_2} p = (1 - p)^{x_1 + x_2} p^2$$

for $(x_1, x_2) \in \Omega_{X_1,X_2} = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$. By the definition of expectation

$$E[\sqrt{X_1 X_2}] = \sum_{(x_1,x_2) \in \Omega_{X_1,X_2}} \sqrt{x_1 x_2} \cdot (1 - p)^{x_1 + x_2} p^2.$$

8. Continuous joint density

The joint density of X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of W and V is given by

$$f_{W,V}(w, v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent? Are W and V independent?

Solution:

For two random variables X, Y to be independent, we must have $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x \in \Omega_X,\ y \in \Omega_Y$. Let's start with X and Y by finding their marginal PDFs. By definition, and using the fact that the joint PDF is 0 outside of $y > 0$, we get:

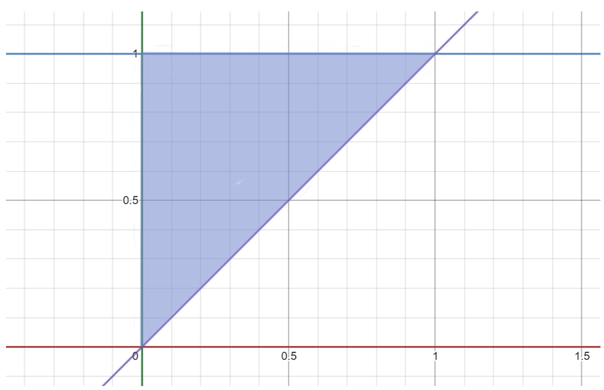
$$f_X(x) = \int_0^\infty xe^{-(x+y)}dy = e^{-x}x$$

We do the same to get the PDF of Y, again over the range $x > 0$:

$$f_Y(y) = \int_0^\infty xe^{-(x+y)}dx = e^{-y}$$

Since $e^{-x}x \cdot e^{-y} = xe^{-x-y} = xe^{-(x+y)}$ for all $x, y > 0$, X and Y are independent.

We can see that W and V are not independent simply by observing that $\Omega_W = (0,1)$ and $\Omega_V = (0,1)$, but $\Omega_{W,V}$ is not equal to their Cartesian product. Specifically, looking at their range of $f_{W,V}(w,v)$. Graphing it with w as the "x-axis" and v as the "y-axis", we see that :



The shaded area is where the joint pdf is strictly positive. Looking at it, we can see that it is not rectangular, and therefore it is not the case that $\Omega_{W,V} = \Omega_W \times \Omega_V$. Remember, the joint range being the Cartesian product of the marginal ranges is not sufficient for independence, but it is *necessary*. Therefore, this is enough to show that they are not independent.

9. Trapped Miner

A miner is trapped in a mine containing 3 doors.

- D_1: The 1^{st} door leads to a tunnel that will take him to safety after 3 hours.
- D_2: The 2^{nd} door leads to a tunnel that returns him to the mine after 5 hours.
- D_3: The 3^{rd} door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters $(12, \frac{1}{3})$.

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

Solution:

Let T = number of hours for the miner to reach safety. (T is a random variable)
Let D_i be the event the i^{th} door is chosen. $i \in \{1, 2, 3\}$. Finally, let T_3 be the time it takes to return to the mine in the third case only (a random variable). Note that the expectation of T_3 is $12 * \frac{1}{3}$ because it is binomially distributed with parameters $n = 12, p = \frac{1}{3}$. By Law of Total Expectation, linearity of expectation, and by applying the conditional expectations given by the problem statement:

$$\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[T|D_1]\,\mathbb{P}(D_1) + \mathbb{E}[T|D_2]\,\mathbb{P}(D_2) + \mathbb{E}[T|D_3]\,\mathbb{P}(D_3) \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3 + T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3] + \mathbb{E}[T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (4 + \mathbb{E}[T]) \cdot \frac{1}{3}
\end{aligned}$$

Solving this equation for $\mathbb{E}[T]$, we get

$$\mathbb{E}[T] = 12$$

Therefore, the expected number of hours for this miner to reach safety is 12.

10. Lemonade Stand

Suppose I run a lemonade stand, which costs me \$100 a day to operate. I sell a drink of lemonade for \$20. Every person who walks by my stand either buys a drink or doesn't (no one buys more than one). If it is raining, n_1 people walk by my stand, and each buys a drink independently with probability p_1. If it isn't raining, n_2 people walk by my stand, and each buys a drink independently with probability p_2. It rains each day with probability p_3, independently of every other day. Let X be my profit over the next week. In terms of n_1, n_2, p_1, p_2 and p_3, what is $\mathbb{E}[X]$?

Solution:

Let R be the event it rains. Let X_i be how many drinks I sell on day i for $i = 1, ..., 7$. We are interested in $X = \sum_{i=1}^{7} (20X_i - 100)$. We have $X_i|R \sim \textsf{Binomial}(n_1, p_1)$, so $\mathbb{E}[X_i|R] = n_1 p_1$. Similarly, $X_i|R^C \sim \textsf{Binomial}(n_2, p_2)$, so $\mathbb{E}\left[X_i|R^C\right] = n_2 p_2$. By the law of total expectation,

$$\mu = \mathbb{E}[X_i] = \mathbb{E}[X_i|R]\,\mathbb{P}(R) + \mathbb{E}\left[X_i|R^C\right]\mathbb{P}(R^C) = n_1 p_1 p_3 + n_2 p_2 (1 - p_3)$$

Hence, by linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{7} (20X_i - 100)\right] &= 20 \sum_{i=1}^{7} \mathbb{E}[X_i] - 700 = 140\mu - 700 \\ &= 140 \cdot (n_1 p_1 p_3 + n_2 p_2 (1 - p_3)) - 700.\end{aligned}$$

11. 3 points on a line

Three points X_1, X_2, X_3 are selected at random on a line L (continuous independent uniform distributions). What is the probability that X_2 lies between X_1 and X_3? **Solution:**

Let $X_1, X_2, X_3 \sim Unif(0, 1)$.

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < X_2 < X_3 \mid X_2 = x)\; f_{X_2}(x)\; dx && \text{Continuous LoTP} \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x, X_3 > x)\; f_{X_2}(x)\; dx && \text{Independence of } X_1, X_2, X_3 \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x)\; \mathbb{P}(x < X_3)\; f_{X_2}(x)\; dx && \text{Independence of } X_1, X_3 \\
&= \int_{-\infty}^{\infty} F_{X_1}(x)\; (1 - F_{X_3}(x))\; f_{X_2}(x)\; dx \\
&= \int_{0}^{1} x\; (1 - x)\; 1\; dx \\
&= \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}
\end{aligned}$$