CSE 312 Section 8

Tail Bounds, Joint Distributions, Law of Total Expectation

Announcements & Reminders

- HW6
  - Due on Wednesday 2/26
  - Late deadline Saturday 3/1 @ 11:59 pm

- HW7
  - Released
  - Due Wednesday 3/5 @ 11:59 pm
  - Late deadline Saturday 3/8 @ 11:59 pm

Review & Questions

Any lingering questions from this last week?

Each week in section, we'll be reviewing the main concepts from this week and putting them into action by going through some practice problems together. But before we get into that review, we'll try to start off each section with some time for you to ask questions. Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Review of Main Concepts

- **Multivariate: Discrete to Continuous:**

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Independence must have	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

Review of Main Concepts

- **Law of Total Probability (r.v. version):** If X is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A|X = x) p_X(x) \qquad \text{discrete } X$$

- **Law of Total Expectation (Event Version)**: Let X be a discrete random variable, and let events $A_1, ..., A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X|A_i] \, \mathbb{P}(A_i)$$

- **Conditional Expectation**: See table. Note that linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y|A] = \mathbb{E}[X|A] + \mathbb{E}[Y|A]$
- **Law of Total Expectation (RV Version)**: Suppose X and Y are random variables. Then,

$$\mathbb{E}[X] = \sum_{y} \mathbb{E}[X|Y = y] \, p_Y(y) \qquad \text{discrete version.}$$

Review of Main Concepts

- **Conditional distributions**

	Discrete	Continuous
Conditional PMF/PDF	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X\|Y=y] = \sum_x x p_{X\|Y}(x\|y)$	$\mathbb{E}[X\|Y=y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$

- **Continuous Law of Total Probability:**

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A|X=x) f_X(x) dx$$

- **Continuous Law of Total Expectation:**

$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X|Y=y] f_Y(y) dy$$

Review of Main Concepts

- **Markov's Inequality**: Let X be a non-negative random variable, and $\alpha > 0$. Then,

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

- **Chebyshev's Inequality**: Suppose Y is a random variable with $\mathbb{E}[Y] = \mu$ and $\mathrm{Var}(Y) = \sigma^2$. Then, for any $\alpha > 0$,

$$\mathbb{P}(|Y - \mu| \geq \alpha) \leq \frac{\sigma^2}{\alpha^2}$$

- **(Multiplicative) Chernoff Bound**: Let $X_1, X_2, ..., X_n$ be *independent* Bernoulli random variables.

  Let $X = \Sigma_{i=1}^n X_i$, and $\mu = \mathbb{E}[X]$. Then, for any $0 \leq \delta \leq 1$,

  - $\mathbb{P}(X \geq (1 + \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}$
  - $\mathbb{P}(X \leq (1 - \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$

Review Questions

Question #1

a) True or False: Markov's Inequality always gives a non-negative result.

- True
- False

Question #2

b) Suppose C and D are discrete random variables. Then $E[C|D = d] =$

- $\sum_d d \cdot p_{D|C}\,(d|c)$
- $\sum_c c \cdot p_{D|C}(d|c)$
- $\int_{-\infty}^{\infty} c f_{c|d} dx$
- $\frac{E[C]}{E[D]}$

Question #3

c) Suppose X and Y are random variables and A is an event. Given that $E[X|A] = 4$ and $E[Y|A] = 10$, what is $E[2X + \frac{Y}{2}|A]$

- 14
- 18
- 9
- 13

Question #4

d) True or false: Chebyshev's Inequality can best be described as giving an upper bound on the distribution's right tail.

- True
- False

Problem 6 –
Do You “Urn” to Learn More About Probability

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- Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the i-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

a) X_1, X_2
b) X_1, X_2, X_3

Work on this with the people around you and then we’ll go over it together!

Problem 6 – Do You “Urn” to Learn More About Probability

a) X_1, X_2

Problem 6 – Do You "Urn" to Learn More About Probability

b) X_1, X_2, X_3

Problem 3 – Exponential Tail Bounds

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Let $X \sim \text{Exp}(\lambda)$ and $k > \frac{1}{\lambda}$.

a) Use Markov's inequality to bound $P(X \geq k)$.
b) Use Markov's inequality to bound $P(X < k)$.
c) Use Chebyshev's inequality to bound $P(X \geq k)$.
d) What is the exact formula for $P(X \geq k)$.
e) For $\lambda k \geq 3$, how do the bounds given in parts a, c, and d compare?

Work on this with the people around you and then we'll go over it together!

Problem 3 – Exponential Tail Bounds

a) Use Markov's inequality to bound $P(X \geq k)$.

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b) Use Markov's inequality to bound $P(X < k)$.

Problem 3 – Exponential Tail Bounds

c) Use Chebyshev's inequality to bound $P(X \geq k)$.

Problem 3 – Exponential Tail Bounds

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Problem 3 – Exponential Tail Bounds

e) For $\lambda k \geq 3$, how do the bounds given in parts a, c, and d compare?

Problem 10 – Lemonade Stand

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- Suppose I run a lemonade stand, which costs me \$100 a day to operate. I sell a drink of lemonade for \$20. Every person who walks by my stand either buys a drink or doesn't (no one buys more than one). If it is raining, n_1 people walk by my stand, and each buys a drink independently with probability p_1. If it isn't raining, n_2 people walk by my stand, and each buys a drink independently with probability p_2. It rains each day with probability p_3, independently of every other day. Let X be my profit over the next week. In terms of $n_1, n_2, p_1, p_2,$ and p_3, what is $E[X]$?

Work on this with the people around you and then we'll go over it together!

Problem 9 – Trapped Miner

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A miner is trapped in a mine containing 3 doors.

- D_1: The 1st door leads to a tunnel that will take him to safety after 3 hours.
- D_2: The 2nd door leads to a tunnel that returns him to the mine after 5 hours.
- D_3: The 3rd door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters $(12, \frac{1}{3})$.

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

Problem 8 – Continuous Joint Density

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The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

And the joint density of W and V is given by

$$f_{W,V}(w,v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent? Are W and V independent?

That’s All, Folks!

Thanks for coming to section this week!
Any questions?