CSE 312 Section 6

Continuous RVs and Midterm Reviews

Administrivia

Announcements & Reminders

- Midterm Exam
  - Next Wednesday 2/19
  - More information has been posted on the course website

- HW5
  - Last homework before the midterm
  - Due Friday 2/14 @ 11:59pm
  - Late deadline Sunday 2/16 @ 11:59pm

Review & Questions

Any lingering questions from this last week?

Each week in section, we’ll be reviewing the main concepts from this week and putting them into action by going through some practice problems together. But before we get into that review, we’ll try to start off each section with some time for you to ask questions. Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Kahoot for content review!

see task 1 from section handout

Review of Main Concepts

- **Cumulative Distribution Function (CDF)**: For any random variable X, the CDF is defined as $F_X(x) = \mathbb{P}(X \le x)$.
  - Notice the CDF is monotonic (non-decreasing), i.e. if $x < y$ then $F_X(x) \le F_X(y)$
  - The CDF is bounded between 0 and 1.
- A **Continuous RV** is one for which the CDF is continuous everywhere.
  - Support is an uncountably infinite set.
- **Probability Density Function (PDF)** is defined as $f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x)$
  - Implies that $F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\mathrm{d}t$
  - $\mathbb{P}(a \le X \le b) = \int_a^b f_X(t)\mathrm{d}t$
  - $f_X(x) \ge 0$

Review of Main Concepts

- **I.I.D**: Random variables are "independent and identically distributed" if they are independent and have the same PDF/PMF
- For continuous random variables:
  - $\mathbb{P}(X = x) = 0 \neq f_X(x)$
  - $F_X(x) = \int_{-\infty}^{x} f_X(t)\mathrm{d}t$
  - $\int_{-\infty}^{\infty} f_X(t)\mathrm{d}t = 1$
  - $\mathbb{E}[X] = \int_{-\infty}^{\infty} t \cdot f_X(t)\mathrm{d}t$
  - $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(t) \cdot f_X(t)\mathrm{d}t$

Problem 2 – Uniform2

2 – Uniform2

Robbie decided he wanted to create a "new" type of distribution that will be famous, but he needs some help. He knows he wants it to be continuous and have uniform density, but he needs help working out some of the details. We'll denote a random variable X having the "Uniform-2" distribution as $X \sim Uniform2(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

(a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\ \infty)$. (Hint: use a piecewise definition).

(b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\ \infty)$. (Hint: use a piecewise definition).

Work on this problem with the people around you, and then we'll go over it together!

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(a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\ \infty)$. (Hint: use a piecewise definition).

$$f_X(x) = \begin{cases} \dfrac{1}{(b - a) + (d - c)}, & x \in [a, b] \cup [c, d] \\ 0, & otherwise \end{cases}$$

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(b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\ \infty)$. (Hint: use a piecewise definition).

$$F_X(x) = \begin{cases} 0, & x \in (-\infty, a) \\ \dfrac{(x - a)}{(b - a) + (d - c)}, & x \in [a, b) \\ \dfrac{(b - a)}{(b - a) + (d - c)}, & x \in [b, c) \\ \dfrac{(b - a) + (x - c)}{(b - a) + (d - c)}, & x \in [c, d) \\ 1, & x \in [d, \infty) \end{cases}$$

Problem 6 – Throwing a dart

6 – Throwing a dart

Consider the closed unit circle of radius r, i.e. S={ $(x, y) : x^2 + y^2 \le r^2$ }. Suppose we throw a dart onto this circle and are guaranteed to hit it, but the dart is equally likely to land anywhere in S. Concretely, this means that the probability that the dart lands in any particular area of size A is equal to $\frac{A}{Area\ of\ the\ whole\ circle}$. The density outside the unit circle is 0.

Let X be the distance the dart lands from the center. What is the CDF and PDF of X? What is $E[X]$ and $Var(X)$?

Work on this problem with the people around you, and then we'll go over it together!

6 – Throwing a dart

Since $F_X(x)$ is the probability that the dart lands inside the circle of radius x, that probability is the area of a circle of radius x divided by the area of the circle of radius r (i.e., $\frac{\pi \cdot x^2}{\pi \cdot r^2}$). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/r^2 & 0 \leq x \leq r \\ 1 & x > r \end{cases}$$

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$$f_X(x) \begin{cases} 2x/r^2 & 0 < x \le r \\ 0 & otherwise \end{cases}$$

Using the definition of expectation we get:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^r \frac{2x^2}{r^2} dx = \frac{2}{3r^2}\left(x^3 \Big|_0^r\right) = \frac{2}{3} r$$

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We know that $Var(X) = E[X^2] - E[X]^2$:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^r \frac{2x^3}{r^2} dx = \frac{2}{4r^2}\left(x^4 \Big|_0^r\right) = \frac{1}{2} r^2$$

Plugging this into our variance equation gives:

$$\Rightarrow Var(X) = \frac{1}{2} r^2 - \left(\frac{2}{3} r\right)^2 = \frac{1}{18} r^2$$

Midterm Review

Topic Coverage

- Counting
- Pigeonhole principle
- Stars and bars
- Binomial theorem
- Inclusion-exclusion
- Probability
- Conditional probability and Bayes Rules
- Independence
- Discrete RV (defining RVs, PMF, CDF, expectation, variance)
- Linearity of expectation
- Discrete RV Zoo

Problem 5 (Midterm Review)

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How many integers in $\{1,2,...,360\}$ are divisible by one or more of the numbers 2, 3, and 5?

Work on this problem with the people around you, and then we'll go over it together!

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Inclusion-exclusion:

$$\frac{360}{2}+\frac{360}{3}+\frac{360}{5}-\frac{360}{2\times 3}-\frac{360}{2\times 5}-\frac{360}{3\times 5}+\frac{360}{2\times 3\times 5}=80+120+72-60-36-24+12=264$$

Problem 3 (Midterm Review)

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Consider the following inequality: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \leq 70$. A solution to this inequality over the nonnegative integers is a choice of a nonnegative integer for each of the 6 variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the inequality. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the inequality?

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This is equivalent to asking how many different solutions are there to the equation $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 70$.

Using stars and bars, we see that the answer is $\binom{76}{6}$

Problem 12 (Midterm Review)

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You are working on a difficult passage from a new piece you are learning on the piano. You wish to play it correctly 4 times before stopping for the day. If your probability of playing it correctly on each attempt is 2/3, and the attempts are independent (unfortunately!), what is the probability that you have to play it at least 8 times?

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This is equivalent to the probability that, in the first 7 attempts, you play it correctly 3 or fewer times. Let X be the number of times you play it correctly in the first 7 attempts. Then $X \sim Bin(7, \frac{2}{3})$

$$\mathbb{P}(X \leq 3) = \binom{7}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^7 + \binom{7}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^6 + \binom{7}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^5 + \binom{7}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^4$$

$$= \frac{379}{3^7} \approx 0.173$$

Problem 10 (Midterm Review)

10 (Midterm Review)

Suppose you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur? Assume that there are 365 possible birthdays and each one is equally probable for a randomly chosen person.

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This is a geometric distribution:

$$\left(\frac{364}{365}\right)^{19} \frac{1}{354} \approx 0.0026$$

Problem 9 (Midterm review)

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The space shuttle has 6 O-rings: these were involved in the Challenger disaster. When the space shuttle is launched, each O-ring has a probability of failure of 0.0137, independent of whether other O-rings fail.

(a) What is the probability that, during 23 launches, no O-ring will fail, but that at least one O-ring will fail during the 24th launch?

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(a) What is the probability that, during 23 launches, no O-ring will fail, but that at least one O-ring will fail during the 24th launch?

The probability that no O-ring fails on a single launch is $(1 - 0.0137)^6 \approx 0.921$. The probability that this happens for 23 launches and doesn't happen on the 24^{th} launch is $0.921^{23}(1 - 0.921) \approx 0.0118$.

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$$0.921^{24} \approx 0.137$$

Problem 7 (Midterm review)

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You are trying to diagnose the probability that a patient with a positive blood sugar test result has diabetes, even though she is in a low-risk group. The probability of a woman in this group having diabetes is 0.8%. 90% of women with diabetes will test positive in the blood sugar test. 7% of women without diabetes will test positive in the blood sugar test. Your patient tests positive in the blood sugar test. What is the probability that she has diabetes?

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Let D be the event that she has diabetes and + be the event of a positive test.

$$\mathbb{P}(D \mid +) = \frac{\mathbb{P}(+ \mid D)\mathbb{P}(D)}{\mathbb{P}(+ \mid D)\mathbb{P}(D) + \mathbb{P}(+ \mid \bar{D})\mathbb{P}(\bar{D})} = \frac{0.9 \times 0.008}{0.9 \times 0.008 + 0.07 \times 0.992} \approx 0.09$$

Problem 1 (Midterm review)

1

Let A and B be events in the same sample space that each have nonzero probability. For each of the following statements, state whether it is always true, always false, or it depends on information not given.

a) If A and B are mutually exclusive, then they are independent.

b) If A and B are independent, then they are mutually exclusive.

c) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are mutually exclusive.

d) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are independent.

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Depends whether $\mathbb{P}(A \cap B) = 9/16$

Problem 2 (Midterm review)

2

Given any set of 18 integers, show that one may always choose two of them so that their difference is divisible by 17.

Work on this problem with the people around you, and then we'll go over it together!

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Given any set of 18 integers, show that one may always choose two of them so that their difference is divisible by 17.

By the pigeonhole principle, two of them, say x and y, must have the same remainder when divided by 17. That means $x \equiv y \pmod{17}$, which in turn means 17 divides $x - y$.

Problem 4 (Midterm Review)

4

You roll three fair dice, each with a different numbers of faces: die 1 has six faces (numbered $1 \ldots 6$), die 2 has eight faces (numbered $1 \ldots 8$), and die 3 has twelve faces (numbered $1 \ldots 12$). Let the random variable X be the sum of the three values rolled. What is $\mathbb{E}[X]$?

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Let D_1, D_2, D_3 be the values of die 1, die 2, and die 3, respectively.

$\mathbb{E}[D_1] = 3.5$
$\mathbb{E}[D_2] = 4.5$
$\mathbb{E}[D_3] = 6.5$

Therefore,

$$\mathbb{E}[X] = \mathbb{E}[D_1 + D_2 + D_3] = \mathbb{E}[D_1] + \mathbb{E}[D_2] + \mathbb{E}[D_3] = 3.5 + 4.5 + 6.5 = 14.5$$

Problem 3 – Create the distribution

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- Suppose X is a continuous random variable that is uniform on $[0, 1)$ and uniform on $[1, 2]$, but

$$\mathbb{P}(1 \leq X \leq 2) = 2 \cdot \mathbb{P}(0 \leq X \leq 1)$$

Outside of $[0, 2]$, the density is 0. What is the PDF and CDF of X?

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The fact that X is uniform on each of the intervals means that its PDF is constant on each. So,

$$f_X(x) = \begin{cases} c, & 0 \leq x < 1 \\ d, & 1 \leq x \leq 2 \\ 0, & otherwise \end{cases}$$

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Note that $F_X(1) - F_X(0) = c$ and $F_X(2) - F_X(1) = d$.

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Note that $F_X(1) - F_X(0) = c$ and $F_X(2) - F_X(1) = d$.

The area under the PDF must be 1, so

$$1 = F_X(2) - F_X(0) = F_X(2) - F_X(1) + F_X(1) - F_X(0) = d + c$$

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Additionally,

$$d = F_X(2) - F_X(1) = P(1 \leq X \leq 2) = 2P(0 \leq X \leq 1) = 2\big(F_X(1) - F_X(0)\big) = 2c$$

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To solve for c and d in our PDF, we need only solve the system of two equations from above: $d = 2c, d + c = 1$ So, $d = \frac{2}{3}, c = \frac{1}{3}$.

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$$d = F_X(2) - F_X(1) = P(1 \le X \le 2) = 2P(0 \le X \le 1) = 2\big(F_X(1) - F_X(0)\big) = 2c$$

To solve for c and d in our PDF, we need only solve the system of two equations from above: $d = 2c, d + c = 1$ So, $d = \frac{2}{3}, c = \frac{1}{3}$. Taking the integral of the PDF yields the CDF, which looks like

$$f_X(x) = \begin{cases} 1/3, & 0 \le x < 1 \\ 2/3, & 1 \le x \le 2 \\ 0, & otherwise \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{3}, & 0 \le x < 1 \\ \frac{2x}{3} - \frac{1}{3}, & 1 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

Problem 4 – Max of uniforms

4 – Max of uniforms

Let $U_1, U_2, \dots, U_n$ be mutually independent uniform random variables on $(0,1)$. Find the CDF and PDF for the random variable $Z = \max(U_1, \dots, U_n)$.

Work on this problem with the people around you, and then we'll go over it together!

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The key idea for solving this question is realizing that the max of n numbers $\max(a_1, \dots, a_n)$ is less than some constant c, iff each individual number is less than that constant c (i.e. $a_i < c$ for all i). Using this idea, we get

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So, the CDF of Z is then

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Finding the PDF is just taking the derivative:

$$f_Z(x) = \begin{cases} n \cdot x^{n-1}, & 0 \le x \le 1 \\ 0, & otherwise \end{cases}$$

That’s All, Folks!

Thanks for coming to section this week!
Any questions?