Section 5: Important Discrete Distributions, More practice with r.v.s

Review of Main Concepts

- **Independence**: Random variable X and event E are independent iff

$$\forall x, \quad \mathbb{P}(X = x \cap E) = \mathbb{P}(X = x)\mathbb{P}(E)$$

  Random variables X and Y are independent iff

$$\forall x \forall y, \quad \mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

  In this case, we have $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ (the converse is not necessarily true).

- **i.i.d. (independent and identically distributed)**: Random variables $X_1, \ldots, X_n$ are i.i.d. (or iid) iff they are mutually independent and have the same probability mass function.

- **Independence of functions of a r.v.**: If X and Y are independent and $g(\cdot), h(\cdot)$ are functions mapping real numbers to real numbers, then $g(X)$ and $h(Y)$ are independent. (See if you can prove this!)

- **Variance of Independent Variables**: If X is independent of Y, $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$. This depends on independence, whereas linearity of expectation always holds. Note that this combined with the above shows that $\forall a, b, c \in \mathbb{R}$ and if X is independent of Y, $Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$.

- Review: Zoo of Discrete Random Variables

  (a) **Uniform**: $X \sim \mathsf{Uniform}(a, b)$ ($\mathsf{Unif}(a, b)$ for short), for integers $a \leq b$, iff X has the following probability mass function:

$$p_X(k) = \frac{1}{b - a + 1}, \quad k = a, a + 1, \ldots, b$$

  $\mathbb{E}[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)(b-a+2)}{12}$. This represents each integer from $[a, b]$ to be equally likely. For example, a single roll of a fair die is $\mathsf{Uniform}(1, 6)$.

  (b) **Bernoulli (or indicator)**: $X \sim \mathsf{Bernoulli}(p)$ ($\mathsf{Ber}(p)$ for short) iff X has the following probability mass function:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1 - p, & k = 0 \end{cases}$$

  $\mathbb{E}[X] = p$ and $Var(X) = p(1 - p)$. An example of a Bernoulli r.v. is one flip of a coin with $\mathbb{P}(\text{head}) = p$.

  (c) **Binomial**: $X \sim \mathsf{Binomial}(n, p)$ ($\mathsf{Bin}(n, p)$ for short) iff X is the sum of n iid $\mathsf{Bernoulli}(p)$ random variables. X has probability mass function

$$p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n$$

  $\mathbb{E}[X] = np$ and $Var(X) = np(1 - p)$. An example of a Binomial r.v. is the number of heads in n independent flips of a coin with $\mathbb{P}(\text{head}) = p$. Note that $\mathsf{Bin}(1, p) \equiv \mathsf{Ber}(p)$. As $n \to \infty$ and $p \to 0$, with $np = \lambda$, then $\mathsf{Bin}(n, p) \to \mathsf{Poi}(\lambda)$. If $X_1, \ldots, X_n$ are independent Binomial r.v.'s, where $X_i \sim \mathsf{Bin}(N_i, p)$, then $X = X_1 + \ldots + X_n \sim \mathsf{Bin}(N_1 + \ldots + N_n, p)$.

  (d) **Geometric:** $X \sim \mathsf{Geometric}(p)$ ($\mathsf{Geo}(p)$ for short) iff X has the following probability mass function:

$$p_X(k) = (1 - p)^{k-1} p, \quad k = 1, 2, \ldots$$

  $\mathbb{E}[X] = \frac{1}{p}$ and $Var(X) = \frac{1-p}{p^2}$. An example of a Geometric r.v. is the number of independent coin flips up to and including the first head, where $\mathbb{P}(\text{head}) = p$.

(e) **Poisson**: $X \sim \mathsf{Poisson}(\lambda)$ ($\mathsf{Poi}(\lambda)$ for short) iff X has the following probability mass function:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

$\mathbb{E}[X] = \lambda$ and $Var(X) = \lambda$. An example of a Poisson r.v. is the number of people born during a particular minute, where λ is the average birth rate per minute. If $X_1, \ldots, X_n$ are independent Poisson r.v.'s, where $X_i \sim \mathsf{Poi}(\lambda_i)$, then $X = X_1 + \ldots + X_n \sim \mathsf{Poi}(\lambda_1 + \ldots + \lambda_n)$.

(f) **Negative Binomial**: $X \sim \mathsf{NegativeBinonial}(r, p)$ ($\mathsf{NegBin}(r, p)$ for short) iff X is the sum of r iid $\mathsf{Geometric}(p)$ random variables. X has probability mass function

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots$$

$\mathbb{E}[X] = \frac{r}{p}$ and $Var(X) = \frac{r(1-p)}{p^2}$. An example of a Negative Binomial r.v. is the number of independent coin flips up to and including the r^{th} head, where $\mathbb{P}(\text{head}) = p$. If $X_1, \ldots, X_n$ are independent Negative Binomial r.v.'s, where $X_i \sim \mathsf{NegBin}(r_i, p)$, then $X = X_1 + \ldots + X_n \sim \mathsf{NegBin}(r_1 + \ldots + r_n, p)$.

(g) **Hypergeometric**: $X \sim \mathsf{HyperGeometric}(N, K, n)$ ($\mathsf{HypGeo}(N, K, n)$ for short) iff X has the following probability mass function:

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad k = \max\{0, n+K-N\}, \ldots, \min\{K, n\}$$

$\mathbb{E}[X] = n\frac{K}{N}$. This represents the number of successes drawn, when n items are drawn from a bag with N items (K of which are successes, and $N - K$ failures) without replacement. If we did this with replacement, then this scenario would be represented as Bin $\left(n, \frac{K}{N}\right)$.

1. Content Review Questions

(a) True or false: $Var(A + B) = Var(A) + Var(B)$

(b) What is $Var(3A + 4)$?

- ◯ $3Var(A) + 4$
- ◯ $3Var(A)$
- ◯ $9Var(A)$
- ◯ $Var(A)$

(c) True or false: $\mathbb{E}[A + B] = \mathbb{E}[A] + \mathbb{E}[B]$

(d) What is $\mathbb{E}[3A + 4]$?

- ◯ $3\mathbb{E}[A] + 4$
- ◯ $3\mathbb{E}[A]$
- ◯ $9\mathbb{E}[A]$
- ◯ $\mathbb{E}[A]$

2. Pond Fishing

Suppose I am fishing in a pond with B blue fish, R red fish, and G green fish, where $B + R + G = N$. Each fish is equally likely to be caught. For each of the following scenarios, identify the most appropriate distribution (with parameter(s)):

(a) How many of the next 10 fish I catch are blue, if I catch and release

- ◯ $$\text{Bin}\left(10, \frac{B}{N}\right)$$
- ◯ $$\text{Ber}\left(\frac{B}{N}\right)$$
- ◯ $$\text{Bin}\left(1, \frac{B}{N}\right)$$

(b) How many fish I had to catch until my first green fish, if I catch and release

- ◯ $$\text{Ber}\left(\frac{G}{N}\right)$$
- ◯ $$\text{Bin}\left(1, \frac{G}{N}\right)$$
- ◯ $$\text{Geo}\left(\frac{G}{N}\right)$$

(c) How many red fish I catch in the next five minutes, if I catch on average r red fish per minute

- ◯ $$\text{Poi}(5R)$$
- ◯ $$\text{Bin}\left(5, \frac{R}{N}\right)$$
- ◯ $$\text{Poi}(5r)$$

(d) Whether or not my next fish is blue

- ◯ $$\text{Poi}(5B)$$
- ◯ $$\text{Bin}\left(1, \frac{R}{N}\right)$$
- ◯ $$\text{Ber}\left(\frac{B}{N}\right)$$

(e) (optional) How many of the next 10 fish I catch are blue, if I do not release the fish back to the pond after each catch

(f) (optional) How many fish I have to catch until I catch three red fish, if I catch and release

3. Balls in Bins

Note: this problem also appeared on the section 4 handout.
Let X be the number of bins that remain empty when m balls are distributed into n bins randomly and independently. For each ball, each bin has an equal probability of being chosen. (Notice that two bins being empty are not independent events: if one bin is empty, that decreases the probability that the second bin will also be empty. This is particularly obvious when $n = 2$ and $m > 0$.) Find $\mathbb{E}[X]$.

4. 3-sided Die

Note: a variation of this problem also appeared on the section 4 handout. Let the random variable X be the sum of two independent rolls of a fair 3-sided die. (If you are having trouble imagining what that looks like, you can use a 6-sided die and change the numbers on 3 of its faces.)

(a) What is the probability mass function of X?

(b) Find $\mathbb{E}[X]$.

(c) What is $Var(X)$?

5. Best Coach Ever!!

You are a hardworking boxer. Your coach tells you that the probability of your winning a boxing match is 0.2 independently of every other match.

(a) How many matches do you expect to fight until you win 10 times and what kind of random variable is this?

(b) You only get to play 12 matches every year. To win a spot in the Annual Boxing Championship, a boxer needs to win at least 10 matches in a year. What is the probability that you will go to the Championship this year and what kind of random variable is the number of matches you win out of the 12?

(c) Let p be your answer to part (b). How many times can you expect to go to the Championship in your 20 year career?

6. Variance of a Product

Let X, Y, Z be independent random variables with means μ_X, μ_Y, μ_Z and variances $\sigma_X^2, \sigma_Y^2, \sigma_Z^2$, respectively. Find $Var(XY - Z)$.

7. True or False?

Identify the following statements as true or false (true means always true). Justify your answer.

(a) For any random variable X, we have $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$.

(b) Let X, Y be random variables. Then, X and Y are independent if and only if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

(c) Let $X \sim \mathsf{Binomial}(n, p)$ and $Y \sim \mathsf{Binomial}(m, p)$ be independent. Then, $X + Y \sim \mathsf{Binomial}(n + m, p)$.

(d) Let $X_1, ..., X_{n+1}$ be independent $\mathsf{Bernoulli}(p)$ random variables. Then, $\mathbb{E}[\sum_{i=1}^n X_i X_{i+1}] = np^2$.

(e) Let $X_1, ..., X_{n+1}$ be independent $\mathsf{Bernoulli}(p)$ random variables. Then, $Y = \sum_{i=1}^n X_i X_{i+1} \sim \mathsf{Binomial}(n, p^2)$.

(f) If $X \sim \mathsf{Bernoulli}(p)$, then $nX \sim \mathsf{Binomial}(n, p)$.

(g) If $X \sim \mathsf{Binomial}(n, p)$, then $\frac{X}{n} \sim \mathsf{Bernoulli}(p)$.

(h) For any two independent random variables X, Y, we have $Var(X - Y) = Var(X) - Var(Y)$.

8. Fun with Poissons

Let $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$, and X and Y are independent.

(a) Show that $X + Y \sim Poisson(\lambda_1 + \lambda_2)$ [This was done in class.]

(b) Show that $P(X = k \mid X + Y = n) = P(W = k)$ where $W \sim Bin(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$

9. Memorylessness

We say that a random variable X is memoryless if $\mathbb{P}(X > k + i \mid X > k) = \mathbb{P}(X > i)$ for all non-negative integers k and i. The idea is that X does not *remember* its history. Let $X \sim Geo(p)$. Show that X is memoryless.

10. Poisson Practice

Seattle averages 3 days with snowfall per year. Suppose the number of days with snowfall follows a Poisson distribution.

(a) What is the probability of getting exactly 5 days of snow in a year?

(b) According to the Poisson model, what is the probability of getting 367 days of snow?