CSE 312 Section 4

Random Variables and Expectation

Administrivia

Announcements & Reminders

- HW2
  - Grades released on gradescope – check your submission to read comments
  - Regrade requests open ~24 hours after grades are released and close after a week
- HW3
  - Written and Coding due today, Thursday 1/30 @ 11:59pm
    - Late deadline Saturday 2/1 @ 11:59pm
  - # late days used for HW3 = Max(late days for written, late days for coding)
    - If you submit one day late for homework but two days late for coding, it counts for two late days total
- HW4
  - Released on the course website
  - Due Wednesday 2/5 @ 11:59pm

Review & Questions

Any lingering questions from this last week?

Each week in section, we'll be reviewing the main concepts from this week and putting them into action by going through some practice problems together. But before we get into that review, we'll try to start off each section with some time for you to ask questions. Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Review of Main Concepts

- **Random Variable (rv):** A numeric function $X : \Omega \to \mathbb{R}$ of the outcome.
- **Range/Support:** The support/range of a random variable X, denoted Ω_X, is the set of all possible values that X can take on.
- **Discrete Random Variable (drv):** A random variable taking on a countable (either finite or countably infinite) number of possible values.
- **Probability Mass Function (pmf)** for a discrete random variable X: a function $p_X : \Omega_X \to [0,1]$ with $p_X(x) = \mathbb{P}(X = x)$ that maps possible values of a discrete random variable to the probability of that value happening, such that $\sum_x p_X(x) = 1$.
- **Cumulative Distribution Function (CDF)** for a random variable X: a function $F_X : \mathbb{R} \to \mathbb{R}$ with $F_X(x) = \mathbb{P}(X \leq x)$

Review of Main Concepts

- **Expectation (expected value, mean, or average):** The expectation of a discrete random variable is defined to be

$$\mathbb{E}[X] = \sum_x x\, p_X(x) = \sum_x x\, \mathbb{P}(X = x)$$

The expectation of a function of a discrete random variable $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_x g(x)\, p_X(x)$$

Review of Main Concepts

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- **Linearity of Expectation:** Let X and Y be random variables, and $a, b, c \in \mathbb{R}$. Then, $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$. Also, for any random variables $X_1, \ldots, X_n$,

$$\mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

Review of Main Concepts

- **Variance:** Let X be a random variable and $\mu = E[X]$. The variance of X is defined to be

  $$Var(X) = \mathbb{E}[(X-\mu)^2]$$

  Notice that since this is an expectation of a non-negative random variable ($(X-\mu)^2$), variance is always nonnegative. With some algebra, we can simplify this to

  $$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

- **Standard Deviation:** Let X be a random variable. We define the standard deviation of X to be the square root of the variance, and denote it $\sigma = \sqrt{Var(X)}$

- **Property of Variance:** Let $a, b \in \mathbb{R}$ and let X be a random variable. Then,

  $$Var(aX+b) = a^2 Var(X)$$

Review of Main Concepts

- **Independence:** Random variables X and Y are independent iff

  $$\forall x \forall y, \mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x)\,\mathbb{P}(Y = y)$$

  In this case, we have $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ (the converse is not necessarily true).

- **i.i.d. (independent and identically distributed):** Random variables $X_1, \dots, X_n$ are i.i.d. (or iid) iff they are independent and have the same probability mass function..

- **Variance of Independent Variables:** If X is independent of Y,

  $$Var(X + Y) = Var(X) + Var(Y)$$

  This depends on independence, whereas linearity of expectation always holds. Note that this combined with the above shows that $\forall a, b, c \in \mathbb{R}$ and if X is independent of Y,

  $$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$$

Review Questions

Question #1

The range of a random variable X is the set of probabilities corresponding to the possible values X can take on.

- True
- False

Question #2

What is the relationship between standard deviation and variance of a random variable X?

- $\sigma = (Var(X))^2$
- $\sigma = Var(X^2)$
- $Var(X^2) = \sigma^2$

Question #3

Let X be the random variable representing the outcome of taking the sum of a 3-dice roll of 6-sided dice. Which function would you use to determine the probability that $X = 7$?

- CDF (cumulative distribution function)
- PMF (probability mass function)

Question #4

Let X be the random variable representing the outcome of taking the sum of a 3-dice roll of 6-sided dice. Which function would you use to determine the probability that $X \leq 7$?

- CDF (cumulative distribution function)
- PMF (probability mass function)

Question #5

A random variable X has the PMF

$$p_X(x) = \begin{cases} \frac{1}{4}, & \text{if } x = -1, \\ \frac{1}{4}, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

What is $E[X]$?

- -1/4
- 3/4
- 1
- 2

Question #6

A random variable X has the PMF

$$p_X(x) = \begin{cases} \frac{1}{4}, & \text{if } x = -1, \\ \frac{1}{4}, & \text{if } x = 0, \\ \frac{1}{2}, & \text{if } x = 2, \\ 0, & \text{otherwise.} \end{cases}$$

What is $Var[X]$?

- $3/4$
- 1
- $((1/4) + 2)$ - $(\frac{3}{4})^2 = 27/16$
- $((1/4) + 2) + (\frac{3}{4})^2 = 45/16$

Problem 4 – Kit Kats Again

4 – Kit Kats Again

We have N candies in the jar.
We have K kit kats in the jar.

We are drawing without replacement until we have k kit kats.

$$k \leq K \leq N$$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?
What is $p_X(n) = \mathbb{P}(X = n)$?

Work on finding the range of X with the people around you!

4 – Kit Kats Again

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Min:
Max:
In between:

4 – Kit Kats Again

We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the last kit kat).

What is $p_X(n) = \mathbb{P}(X = n)$?

Problem 5 – Hungry Washing Machine

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You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

b) Find $\mathbb{E}[X]$ from the definition of expectation.

Work on this problem with the people around you, and then we'll go over it together!

5 – Hungry Washing Machine

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5 – Hungry Washing Machine

b) Find $\mathbb{E}[X]$ from the definition of expectation.

Problem 2 – Identify that Range!

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

a) The sum of two rolls of a six-sided die.

b) The number of lottery tickets I buy until I win it.

c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

e) The number of whole minutes I wait at the bus stop for the next bus.

Work on this problem with the people around you, and then we'll go over it together!

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e) The number of whole minutes I wait at the bus stop for the next bus.

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Problem 7 – Frogger

7 – Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let X be the location of the frog.

a) Find $p_X(k)$, the probability mass function for X.

b) Compute $\mathbb{E}[X]$ from the definition.

Work on this problem with the people around you, and then we'll go over it together!

7 – Frogger

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b) Compute $\mathbb{E}[X]$ from the definition.

That's All, Folks!

Thanks for coming to section this week!
Any questions?