# More Joint Distributions

CSE 312 Winter 25
Lecture 19

# Announcements

Ed's interface for the coding question is broken at the moment.

The underlying code is fine, but Ed won't import a library we need.

Gradescope can run the code properly (or a local installation).

Robbie has a ticket with Ed support to get our environment fixed; in the meantime, the coding question will be due with HW7 not HW6 in case you weren't able to work on it.

Robbie is traveling at the end of this week (at SIGCSE)

TAs will guest lecture on Wednesday/Friday.

Robbie will have access to email but will be slower.

# Multiple Random Variables

# Analogues for continuous

Everything we saw today has a continuous version.

There are "no surprises"– replace pmf with pdf and sums with integrals.

| | **Discrete** | **Continuous** |
|---|---|---|
| **Joint PMF/PDF** | $p_{X,Y}(x,y) = P(X=x, Y=y)$ | $f_{X,Y}(x,y) \neq P(X=x,Y=y)$ |
| **Joint CDF** | $F_{X,Y}(x,y) = \sum_{t \le x} \sum_{s \le y} p_{X,Y}(t,s)$ | $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\,ds\,dt$ |
| **Normalization** | $\sum_x \sum_y p_{X,Y}(x,y) = 1$ | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$ |
| **Marginal PMF/PDF** | $p_X(x) = \sum_y p_{X,Y}(x,y)$ | $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$ |
| **Expectation** | $E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$ | $E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$ |
| **Conditional PMF/PDF** | $p_{X\mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$ | $f_{X\mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$ |
| **Conditional Expectation** | $E[X \mid Y=y] = \sum_x x\, p_{X\mid Y}(x \mid y)$ | $E[X \mid Y=y] = \int_{-\infty}^{\infty} x f_{X\mid Y}(x \mid y)\,dx$ |
| **Independence** | $\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ | $\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ |

# Different dice

Roll two fair dice independently.
Let $U$ be the minimum of the two rolls and $V$ be the maximum

What is $\mathbb{P}(U = 2|V = 3)$?

$$\frac{\mathbb{P}(U=2 \cap V=3)}{\mathbb{P}(V=3)} = \frac{2/16}{5/16} = \frac{2}{5}$$

$$p_{U|V}(2|3) = \frac{2}{5}$$

| $p_{U,V}$ | $U$=1 | $U$=2 | $U$=3 | $U$=4 |
|---|---|---|---|---|
| $V$=1 | 1/16 | 0 | 0 | 0 |
| $V$=2 | 2/16 | 1/16 | 0 | 0 |
| $V$=3 | 2/16 | 2/16 | 1/16 | 0 |
| $V$=4 | 2/16 | 2/16 | 2/16 | 1/16 |

# Different dice

$p_{U,V}(a,b) = p_{V,U}(b,a)$

Find these values

$p_{V|U}(2|1) = \dfrac{2/16}{7/16} = \dfrac{2}{7}$

$p_{U|V}(1|2) = \dfrac{2/16}{3/16} = \dfrac{2}{3}$

$p_{U|V}(4|1) =$

| $p_{U,V}$ | $U=1$ | $U=2$ | $U=3$ | $U=4$ |
|---|---|---|---|---|
| $V=1$ | 1/16 | 0 | 0 | 0 |
| $V=2$ | 2/16 | 1/16 | 0 | 0 |
| $V=3$ | 2/16 | 2/16 | 1/16 | 0 |
| $V=4$ | 2/16 | 2/16 | 2/16 | 1/16 |

# Different dice

Find these values

$$p_{V|U}(2|1) = \frac{p_{V,U}(2,1)}{p_U(1)} = \frac{2/16}{7/16} = \frac{2}{7}$$

$$p_{U|V}(1|2) = \frac{p_{U,V}(1,2)}{p_V(2)} = \frac{2/16}{3/16} = \frac{2}{3}$$

$$p_{U|V}(4|1) = \frac{p_{U,V}(4,1)}{p_V(1)} = \frac{0}{1/16} = 0$$

| $p_{U,V}$ | $U$=1 | $U$=2 | $U$=3 | $U$=4 |
|---|---|---|---|---|
| $V$=1 | 1/16 | 0 | 0 | 0 |
| $V$=2 | 2/16 | 1/16 | 0 | 0 |
| $V$=3 | 2/16 | 2/16 | 1/16 | 0 |
| $V$=4 | 2/16 | 2/16 | 2/16 | 1/16 |

# What about the continuous versions?

In the continuous case, everything is still a density function, not a mass function.

Joint density

Marginal density

Conditional density

Expectations, conditional expectations integrate $x \cdot \text{(cond)density}(x)$

You aren't getting a probability, you're getting a density; have to integrate to get a value.

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# Conditioning on probability 0

We said for discrete spaces, when $\mathbb{P}(B) = 0, \mathbb{P}(A|B)$ is undefined

How can you condition on something that doesn't happen?

Also, how can you have $\mathbb{P}(B)$ in the denominator?

For continuous spaces, we have to use densities to avoid the problem, but we can avoid the problem with densities!

$$\boxed{f_{X|Y}(x|y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\frac{X = x \cap Y = y}{Y = y}$$

$\mathbb{P}(Y = y)$ is $0$, but the density might not be $0$ there so this expression can be defined (and it works!).

If density is 0 for $Y = y$, the conditional density is undefined there.

# A note on independence

The definition of independence says $X, Y$ independent if and only if

$p_{X,Y}(x,y) = p_X(x)p_Y(y)$ or $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ (as appropriate)

There's often a nice shortcut. If $X, Y$ are independent then joint support of $X, Y$ (denoted $\Omega_{X,Y}$) must be $\Omega_X \times \Omega_Y$.

Joint support is $\{(x,y): p_{X,Y}(x,y) > 0\}$.

Often easier to verify <u>dependence</u> when those are different (especially in the continuous case).

But note this is a single implication not an if-and-only-if.

# Continuous definitions and theorems

Conditional expectation:

$$\mathbb{E}[X|\mathrm{Y}=\mathrm{y}] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\ \mathrm{d}x$$

LTE:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \mathbb{E}[X|Y=y] \cdot f_Y(y)\ \mathrm{dy}$$

LTP:

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A|X=x) \cdot f_X(x)\ \mathrm{d}x$$

$X$ is continuous; integrating over all values for $X$ gives the full space

# Covariance

# Covariance

We sometimes want to measure how "intertwined" $X$ and $Y$ are – how much knowing about one of them will affect the other.

If $X$ turns out "big" how likely is it that $Y$ will be "big" how much do they "vary together"

## Covariance

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

## Covariance

$$\mathbf{Cov}(\mathbf{X},\mathbf{Y}) = \mathbb{E}[\underbrace{(X - \mathbb{E}[X])}\underbrace{(Y - \mathbb{E}[Y])}] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

If $X, Y$ go in the same direction

$X > E[X]$ $\quad X - E[X]$ $+$

$Y > E[Y]$ $\quad Y - E[Y]$ $+$

$\Big\} +$

$X < E[X]$ $\quad$ '' $-$

$Y < E[Y]$ $\quad$ '' $-$

$\Big\} +$

If $X, Y$ go in the opposite directions

$X > E[X]$ $\quad X - E[X]$ $-$

$Y < E[Y]$ $\quad Y - E[Y]$ $+$

$\Big\} -$

$X <$ $\quad +$

$Y >$ $\quad -$

$\Big\} -$

# Covariance

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$$

That's consistent with our previous knowledge for independent variables. (for $X, Y$ independent, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$).

You and your friend are playing a game, you flip a coin: if heads you pay your friend a dollar, if tails they pay you a dollar. Let $X$ be your profit and $Y$ be your friend's profit.

What is $\mathrm{Var}(X+Y)$?

Before you calculate, make a prediction. What should it be?

# Covariance

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$X^2$ ?

$X=1 \rightarrow X^2=1$

$X=-1 \rightarrow X^2=1$

$p_{X^2}(k) = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{o/w} \end{cases}$

You and your friend are playing a game, you flip a coin: if heads you pay your friend a dollar, if tails they pay you a dollar. Let $X$ be your profit and $Y$ be your friend's profit.

What is $\mathrm{Var}(X+Y)$? $\mathrm{Cov}(X,Y)$?

$\mathrm{Var}(X) = \mathrm{Var}(Y) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1 - 0^2 = 1$

$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$\mathbb{E}[XY] = \frac{1}{2} \cdot (-1 \cdot 1) + \frac{1}{2}(1 \cdot -1) = -1$

$\mathrm{Cov}(X,Y) = -1 - 0 \cdot 0 = -1.$

$\mathrm{Var}(X+Y) = 1 + 1 + 2 \cdot -1 = 0$

# Covariance, Another example

Let $X$ be a Bernoulli RV with probability $p$ of success.

Let $Y = X$ ($Y$ **<u>is</u>** $X$, not an iid copy, literally the same experiment)

Let $Z = -X$

Let $W$ be an independent Bernoulli, indentically distributed to $X$

Find

$\mathrm{Cov}(X, Y)$, $\mathrm{Cov}(X, Z)$, $\mathrm{Cov}(X, W)$

# Covariance, Another example

Let $X$ be a Bernoulli RV with probability $p$ of success.

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Let $W$ be an independent Bernoulli, indentically distributed to $X$

$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$= (1 \cdot 1 \cdot p + 0 \cdot 0 \cdot [1-p]) - p \cdot p$

$= p - p^2 = p(1-p)$

Hey, that's the variance of $X$. This is a pattern: $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$

# Covariance, Another example

Let $X$ be a Bernoulli RV with probability $p$ of success.

Let $Y = X$ ($Y$ **is** $X$, not an iid copy, literally the same experiment)

Let $Z = -X$

Let $W$ be an independent Bernoulli, indentically distributed to $X$

$\text{Cov}(X, Z) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$= (1 \cdot -1 \cdot p + 0 \cdot -0 \cdot [1-p]) - (p \cdot [-p])$

$= -p - [-p^2] = -p(1-p)$

General pattern: $\text{Cov}(X, -Y) = -\text{Cov}(X, Y)$

# Covariance, Another example

Let $X$ be a Bernoulli RV with probability $p$ of success.

Let $Y = X$ ($Y$ **is** $X$, not an iid copy, literally the same experiment)

Let $Z = -X$

Let $W$ be an independent Bernoulli, indentically distributed to $X$

$\text{Cov}(X, W) = \mathbb{E}[XW] - \mathbb{E}[X]\mathbb{E}[W]$

$= (1 \cdot 1 \cdot p^2 + 1 \cdot 0 \cdot p[1-p] + 0 \cdot 1 \cdot [1-p]p + 0 \cdot 0 \cdot [1-p]^2) - (p \cdot [p])$

$= (p^2) - p^2 = 0$

General pattern: if $X, Y$ independent $\text{Cov}(X, Y) = 0$

# A Few Notes

Covariance is an un-normalized number.

It measures both how intertwined $X, Y$ are and in some sense how much $X, Y$ vary in the first place (if you multiply both $X, Y$ by $2$, the strength of the relationship intuitively is the same, but covariance increases).

If you want just the strength of the relationship, you probably want the "correlation coefficient": $\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$ always between $-1$ and $1$.

Covariance directly measures only "linear" relationships; if $Y$ depends on $X^2$, the covariance might not be as high as you expect.

If dealing with real data, look at a plot to see if you should be looking for a linear relationship in the first place.

# A Continuous-ish Example

Recall from Friday: You will flip $2$ (independent, fair coins). Call the number of heads $X$. Then (independently of the coin flips) draw an exponential random variable $Y$ from the distribution $\text{Exp}(X+1)$.

Let's find the PDF of $Y$.

Let $g_\lambda(x) = \lambda e^{-\lambda x}$, i.e. density for Exp($\lambda$) (for $x \geq 0$)

$f_Y(y) = g_1(y) \cdot \frac{1}{4} + g_2(y) \cdot \frac{1}{2} + g_3(y) \cdot \frac{1}{4}$

$f_Y(y) = \frac{1}{4} e^{-y} + \frac{1}{2} \cdot 2 \cdot e^{-2y} + \frac{1}{4} \cdot 3e^{-3y}$ (for $y \geq 0$)

Notice this isn't an exponential random variable!

# A Continuous-ish Example

Now we can check that expectation…

$$\mathbb{E}[Y] = \int_0^\infty y\left(\frac{1}{4}e^{-y} + \frac{1}{2}\cdot 2\cdot e^{-2y} + \frac{1}{4}\cdot 3e^{-3y}\right)\mathrm{d}y$$

$$= \int_0^\infty y\cdot\frac{1}{4}e^{-y}dy + \int_0^\infty y\, e^{-2y}dy + \int_0^\infty y\frac{1}{4}\cdot 3e^{-3y}dy$$

Integral of $ye^{-y}$ will be 1, since that's the expectation of Exp(1)

$$= \frac{1}{4}\cdot 1 + \frac{1}{2}\int_0^\infty 2y\, e^{-2y}dy + \frac{1}{4}\cdot\int_0^\infty y\, 3e^{-3y}dy$$

Setup for same trick, Exp(2), Exp(3)

$$= \frac{1}{4}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{3} = \frac{1}{4} + \frac{1}{4} + \frac{1}{12} = \frac{7}{12}$$