

Linearity of Expectation

CSE 312 Winter 25
Lecture 11

Outline

Linearity of expectation

Statement

Proof

A whole bunch of examples

Expectation

Expectation

The “expectation” (or “expected value”) of a random variable X is:

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} \underbrace{k \cdot \mathbb{P}(X = k)}_{\text{weight}} \quad \leftarrow$$

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} \underbrace{X(\omega)}_{\text{value}} \cdot \underbrace{\mathbb{P}(\omega)}_{\text{weight}} \quad \leftarrow$$

Intuition: The weighted average of values X could take on.
Weighted by the probability you actually see them.

Linearity of Expectation

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For any two random variables X and Y :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

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Note: X and Y do not have to be independent

Extending this to n random variables, X_1, X_2, \dots, X_n

$$\mathbb{E}[X_1 + X_2 + \dots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n]$$


This can be proven by induction.

Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Proof:

$$\begin{aligned}\mathbb{E}[X + Y] &= \sum_{\omega \in \Omega} \mathbb{P}(\omega) (X(\omega) + Y(\omega)) \\ &= \sum_{\omega \in \Omega} \mathbb{P}(\omega) X(\omega) + \sum_{\omega \in \Omega} \mathbb{P}(\omega) Y(\omega) \\ &= \sum_{\omega \in \Omega} \mathbb{P}(\omega) X(\omega) + \sum_{\omega \in \Omega} \mathbb{P}(\omega) Y(\omega) \\ &= \mathbb{E}[X] + \mathbb{E}[Y]\end{aligned}$$

$$\begin{aligned}a(b+c) \\ ab + ac\end{aligned}$$

Linearity of Expectation

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For any two random variables X and Y :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Constants are also fine:

For real numbers a, b, c

$$\begin{aligned}\mathbb{E}[aX + bY + c] &= \mathbb{E}[aX] + \mathbb{E}[bY + c] \\ &= a\mathbb{E}[X] + b\mathbb{E}[Y] + c\end{aligned}$$

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$
- How many fish do both of you bring on an average day?

Fishy Business

$$Z = X + Y$$

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- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$

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Let Z be the r.v. representing the total number of fish you both catch

$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

$\swarrow \searrow$
X Y

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- You can sell each for \$10 per fish, but you need \$15 (total) for expenses.
What is your average profit?

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- You can sell each for \$10 per fish, but you need \$15 (total) for expenses. What is your average profit?

$$\mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15 = 100 - 15 = 85$$

Coin Tosses

If we flip a coin twice, ^{fair} what is the expected number of heads that come up?

Coin Tosses

If we flip a coin twice, what is the expected number of heads that come up?

Let Y be the r.v. representing the total number of heads

$$p_Y(y) = \begin{cases} \frac{1}{4} & \text{if } \underline{y = 0} \\ \frac{1}{2} & \text{if } \underline{y = 1} \\ \frac{1}{4} & \text{if } \underline{y = 2} \\ 0 & \underline{\text{otherwise}} \end{cases}$$

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$$\mathbb{E}[Y] = \sum_{k \in \Omega_Y} p_Y(k) \cdot k = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

Repeated Coin Tosses

np

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let X be the r.v. representing the total number of heads.

Make a prediction --- what should $\mathbb{E}[X]$ be?

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let X be the r.v. representing the total number of heads.

$$\underline{\mathbb{E}[X]} = \sum_{k=0}^n k \cdot \underline{\mathbb{P}(Y = k)} = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Handwritten notes: $\mathbb{P}(X=k)$ above the sum, and blue brackets under the binomial coefficient and the probability terms.

Ok, but what actually is it?
I don't have intuition for this
formula.

Repeated Coin Tosses

Now what if the probability of flipping a head was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=0}^n k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\&= \sum_{k=1}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\&= \sum_{k=1}^n n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k} \\&= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \\&= np(p + (1-p))^{n-1} = np\end{aligned}$$

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

Binomial Theorem!

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

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No one wants to do proofs like this every time!

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Note: X and Y do not have to be independent

Extending this to n random variables, X_1, X_2, \dots, X_n

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Indicator Random Variables

For any event A , we can define the indicator random variable $\mathbf{1}[A]$ for A

$$\mathbf{1}[A] = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}\mathbb{P}(X = 1) &= \mathbb{P}(A) \\ \mathbb{P}(X = 0) &= 1 - \mathbb{P}(A)\end{aligned}$$

You'll also see notation like:

$\mathbf{1}[A]$, $\mathbf{1}_A$, $\mathbf{1}[\text{some boolean}]$

$$p_X(x) = \begin{cases} \mathbb{P}(A) & \text{if } x = 1 \\ 1 - \mathbb{P}(A) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}\mathbb{E}[X] &= 1 \cdot p_X(1) + 0 \cdot p_X(0) \\ &= p_X(1) = \mathbb{P}(A)\end{aligned}$$

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

What indicators can we define? What 'Booleans' have enough information to combine (add) and solve the problem?

Repeated Coin Tosses (Again)

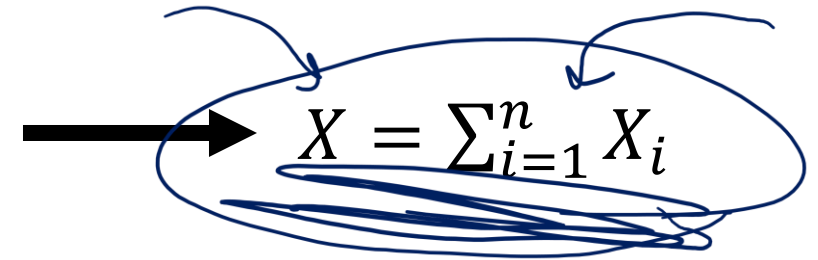
The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

Define X_i as follows:

$$\underline{X_i} = \begin{cases} 1 & \text{if the } i\text{th coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}\mathbb{P}(X_i = 1) &= p \\ \mathbb{P}(X_i = 0) &= 1 - p\end{aligned}$$



A diagram illustrating the relationship between individual coin flip outcomes and the total number of heads. A thick black arrow points from the definition of X_i to the equation $X = \sum_{i=1}^n X_i$. The equation is enclosed in a blue oval, and there are blue scribbles underneath it. Two blue arrows point from the text above to the terms X_i and X in the equation.

$$X = \sum_{i=1}^n X_i$$

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

Repeated Coin Tosses (Again)

The probability of flipping a head is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

$$X = \sum_{i=1}^n X_i$$
$$\mathbb{E}[X_i] = p$$

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\sum_{i=1}^n X_i] \\ &= \mathbb{E}[X_1 + X_2 + \dots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n] \\ &= \sum_{i=1}^n \mathbb{E}[X_i] \\ &= \sum_{i=1}^n p = np\end{aligned}$$

LOE

$$1 \cdot p + 0 \cdot (1-p)$$

p

Computing complicated expectations

We often use these three steps to solve complicated expectations

1. Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + X_2 + \cdots + X_n$$

2. LOE: Apply Linearity of Expectation

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

3. Conquer: Compute the expectation of each X_i

Often X_i are indicator random variables

Pairs with the same birthday

$$\frac{m(m-1)}{2} \text{ pairs}$$

In a class of m students, on average how many pairs of people have the same birthday?

Decompose:

→ 1/365 chance

LOE:

Conquer:

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose: Let X be the number of pairs with the same birthday

Define X_{ij} as follows:

$$\underline{X_{ij}} = \begin{cases} 1 & \text{if person } i, j \text{ have the same birthday} \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{X} = \sum_{i,j} X_{ij}$$

LOE:

Conquer:

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LOE:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i,j} X_{ij}\right] = \sum_{i,j} \mathbb{E}[X_{ij}]$$

Conquer:

Pairs with the same birthday

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LOE:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i,j} X_{ij}\right] = \sum_{i,j} \mathbb{E}[X_{ij}]$$

Conquer:

$$\begin{aligned} \mathbb{E}[X_{ij}] &= \mathbb{P}(X_{ij} = 1) = \frac{365}{365 \cdot 365} = \frac{1}{365} \\ \mathbb{E}[X] &= \binom{m}{2} \cdot \mathbb{E}[X_{ij}] = \binom{m}{2} \cdot \frac{1}{365} \end{aligned}$$

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and $n-1$ (equally likely)

Let X be the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

Decompose:

What X_i can we define that have the needed information?

LOE:

Conquer:

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Decompose: Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person } i \text{ sits in front of their own name tag} \\ 0 & \text{otherwise} \end{cases}$$

Note: $X = \sum_{i=1}^n X_i$

LOE:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$$

Conquer:

These X_i are not independent!
That's ok!!

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Rotate the table by a random number k of positions between 1 and $n-1$ (equally likely)

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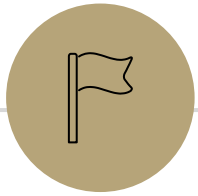
LOE:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$$

Conquer:

$$\mathbb{E}[X_i] = P(X_i = 1) = \frac{1}{n-1}$$

$$\mathbb{E}[X] = n \cdot \mathbb{E}[X_i] = \frac{n}{n-1}$$



Extra Practice

Frogger



A frog starts on a 1-dimensional number line at 0.

Each second, independently, the frog takes a unit step right with probability p_1 , to the left with probability p_2 , and doesn't move with probability p_3 , where $p_1 + p_2 + p_3 = 1$.

After 2 seconds, let X be the location of the frog. **Find $\mathbb{E}[X]$.**

Frogger – Brute Force



A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R , to the left with probability p_L , and doesn't move with probability p_S , where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. Find $\mathbb{E}[X]$.

We could find the PMF by computing the probability for each value in the range of X , and then applying definition of expectation:

$$p_X(x) = \begin{cases} p_L^2 & x = -2 \\ 2p_Lp_S & x = -1 \\ 2p_Lp_R + p_S^2 & x = 0 \\ 2p_Rp_S & x = 1 \\ p_R^2 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

We think about the outcomes that correspond to each value of X and compute the probability of that. For example, $X=0$ happens when the frog is at the same position after 2 sec – this means it either moved left and then right, or right and then left, or did not move both seconds.

$$\mathbb{E}[X] = \sum_{\omega} P(\omega)X(\omega) = (-2)p_L^2 + (-1)2p_Lp_S + 0 \cdot (2p_Lp_R + p_S^2) + (1)2p_Rp_S + (2)p_R^2 = 2(p_R - p_L)$$

Frogger – LOE



Or we can apply LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R , to the left with probability p_L , and doesn't move with probability p_S , where $p_L + p_R + p_S = 1$. After 2 seconds, let X be the location of the frog. Find $\mathbb{E}[X]$.

Define X_i as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^2 X_i\right] = \sum_{i=1}^2 \mathbb{E}[X_i] = 2(p_R - p_L)$$

Frogger – LOE



If we interested in a whole minute (60 sec), the first approach would be awful because we would need to compute many probabilities or deal with a gnarly summation! Instead, we can use LoE!

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_R , to the left with probability p_L , and doesn't move with probability p_S , where $p_L + p_R + p_S = 1$. After 60 seconds, let X be the location of the frog. Find $\mathbb{E}[X]$.

Define X_i as follows:

$$X_i = \begin{cases} -1 & \text{if the frog moved left on the } i\text{th step} \\ 0 & \text{otherwise} \\ 1 & \text{if the frog moved right on the } i\text{th step} \end{cases}$$

$$\mathbb{E}[X_i] = -1 \cdot p_L + 1 \cdot p_R + 0 \cdot p_S = (p_R - p_L)$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{60} X_i\right] = \sum_{i=1}^{60} \mathbb{E}[X_i] = 60(p_R - p_L)$$