Conditional Probability

CSE 312 Winter 25
Lecture 5

Last Time

Sample Space

A sample space Ω is the set of all possible outcomes of an experiment.

Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of Ω)

Probability

A probability is a number between 0 and 1 describing how likely a particular outcome is.

Probability Space

Probability Space

A (discrete) probability space is a pair $(\Omega, \mathbb{P})$ where:

Ω is the sample space

$\mathbb{P}: \Omega \to [0, 1]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(x) \geq 0$ for all x
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$
- If $E, F \subseteq \Omega$ and $E \cap F = \emptyset$ then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

Probability Space

Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Is this a valid probability space?

$\mathbb{P}$ takes in elements of Ω and outputs numbers between 0 and 1

$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$

Uniform Probability Space

The most common probability measure is the **uniform** probability measure. In the uniform measure, for every event E

$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$.

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

A. $\binom{100}{50}/2^{100}$

B. $1/101$

C. $1/2$

D. $1/2^{50}$

E. There is not enough information in this problem.

Mutually Exclusive Events

Two events E, F are mutually exclusive if they cannot happen simultaneously.

In notation, $E \cap F = \emptyset$ (i.e. they're disjoint subsets of the sample space).

For example, if $\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$E_1 =$"the coin came up heads"

$E_2 =$"the coin came up tails"

$E_3 =$"the die showed an even number"

E_1 and E_2 are mutually exclusive.
E_1 and E_3 are not mutually exclusive.

Axioms and Consequences

We wrote down 3 requirements (axioms) on probability measures

- $\mathbb{P}(x) \geq 0$ for all x (non-negativity)
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$ (normalization)
- If E and F are mutually exclusive then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ (countable additivity)

These lead quickly to these three corollaries

- $\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$ (complementation)
- If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$ (monotonicity)
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$ (inclusion-exclusion)

Examples

More Examples!

Suppose you roll two dice. Each die is fair and they don’t affect each other. What is the probability of both dice being even?

What is your sample space?

What is your probability measure $\mathbb{P}$?

What is your event?

What is the probability?

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space? $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

What is your probability measure $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is your event? $\{2,4,6\} \times \{2,4,6\}$

What is the probability? $3^2/6^2$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What if we can't tell the dice apart and always put the dice in increasing order by value.

What is your sample space?

$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(2,5),(2,6)$

$(3,3),(3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)\}$

What is your probability measure $\mathbb{P}$?

$\mathbb{P}((x,y)) = 2/36$ if $x \neq y$, $\mathbb{P}(x,x) = 1/36$

What is your event? $\{(2,2),(4,4),(6,6),(2,4),(2,6),(4,6)\}$

What is the probability? $3 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} = \frac{9}{36}$

Takeaways

There is often more than one sample space possible! But one is probably easier than the others.

Finding a sample space that will make the uniform measure correct will usually make finding the probabilities easier to calculate.

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space

Probability Measure

Event

Probability

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: $\{(x, y): x \text{ and } y \text{ are different cards }\}$

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$

Event: all pairs with equal values

Probability: $\frac{13 \cdot P(4,2)}{52 \cdot 51}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50!}{52!}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50 * 49 * 48 * \cdots . * 2 * 1}{52 * 51 * 50 * 49 * 48 * \ldots * 2 * 1}$

Takeaway

There’s often information you “don’t need” in your sample space.

It won’t give you the wrong answer.

But it sometimes makes for extra work/a harder counting problem,

Good indication: you cancelled A LOT of stuff that was common in the numerator and denominator.

Few notes about events and samples spaces

- If you’re dealing with a situation where you may be able to use a uniform probability space, make sure to set up the sample space in a way that every outcome is equally likely.

- Try not overcomplicate the sample space – only include the information that you need in it.

- When you define an event, make sure it is a <u>subset</u> of the sample space!

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if ?

$\mathbb{P}(E) = 1$ if and only if ?

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if an event can't happen.

$\mathbb{P}(E) = 1$ if and only if an event is guaranteed (every outcome outside E has probability 0).

Conditional Probabilities

Conditioning

You roll a fair red die and a fair blue die (without letting the dice affect each other).

But they fell off the table and you can't see the results.

I can see the results – I tell you the sum of the two dice is 4.

What's the probability that the red die shows a 5, **conditioned** on knowing the sum is 4?

Conditioning

You roll a fair red die and a fair blue die (without letting the dice affect each other).

But they fell off the table and you can't see the results.

I can see the results – I tell you the sum of the two dice is 4.

What's the probability that the red die shows a 5, **conditioned** on knowing the sum is 4?

It's 0.

Without the conditioning it was $1/6$.

Conditioning

When I told you "the sum of the dice is 4" we restricted the sample space.

The only remaining outcomes are $\{(1,3),(2,2),(3,1)\}$ out of $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$.

Outside the (restricted) sample space, the probability is going to become 0. What about the probabilities inside?

Conditional Probability

Conditional Probability

For an event B, with $\mathbb{P}(B) > 0$,
the "Probability of A conditioned on B" is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Just like with the formal definition of probability, this is pretty abstract.

It does accurately reflect what happens in the real world.

If $\mathbb{P}(B) = 0$, we can't condition on it (it can't happen! There's no point in defining probabilities where we know B has not happened) – $\mathbb{P}(A|B)$ is **undefined** when $\mathbb{P}(B) = 0$.

Conditioning…

Let A be "the red die is 5"
Let B be "the sum is 4"
Let C be "the blue die is 3"

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

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$\mathbb{P}(A|B)$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	$\boxed{(1,3)}$	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	$\boxed{(2,2)}$	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	$\boxed{(3,1)}$	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
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Conditioning…

Let A be "the red die is 5"
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$\mathbb{P}(A|B)$

$\mathbb{P}(A \cap B) = \mathbb{P}(\emptyset) = 0$
$\mathbb{P}(B) = 3/36$

$$P(A|B) = \frac{0}{3/36}$$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	$\boxed{(1,3)}$	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	$\boxed{(2,2)}$	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	$\boxed{(3,1)}$	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
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Conditioning…

Let A be "the red die is 5"
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$\mathbb{P}(A|C)$

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Conditioning…

Let A be "the red die is 5"
Let B be "the sum is 4"
Let C be "the blue die is 3"

$\mathbb{P}(A|C)$

$\mathbb{P}(A \cap C) = 1/36$
$\mathbb{P}(C) = 6/36$

$$P(A|C) = \frac{1/36}{6/36}$$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
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Conditioning Practice

Red die 6 conditioned on sum 7

Red die 6 conditioned on sum 9

Sum 7 conditioned on red die 6

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
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D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Take a few minutes to work on this with the people around you! (also on your handout)

Conditioning Practice

A ~ Red die 6

B ~ Sum is 7

$\mathbb{P}(A|B)$

$= \mathbb{P}(A \cap B)/P(B)$

$=$

$= 1/6$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
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D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Conditioning Practice

A ~ Red die 6

C ~ Sum is 9

$\mathbb{P}(A|C)$

$= \mathbb{P}(A \cap C)/P(C)$

$=$

$= 1/4$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
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D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Conditioning Practice

B ~ Sum is 7

A ~ Red die is 6

$\mathbb{P}(B|A)$

$= \mathbb{P}(B \cap A)/P(A)$

$=$

$= 1/6$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
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Conditioning Practice

Red die 6 conditioned on sum 7 1/6

Red die 6 conditioned on sum 9 1/4

Sum 7 conditioned on red die 6 1/6

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
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Direction Matters

Are $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$ the same?

Direction Matters

No! $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$ are different quantities.

$\mathbb{P}($"traffic on the highway" | "it's snowing"$)$ is close to 1

$\mathbb{P}($"it's snowing" | "traffic on the highway"$)$ is much smaller; there many other times when there is traffic on the highway

It's a lot like implications – order can matter a lot!

(but there are some A, B where the conditioning doesn't make a difference)