Wrap Counting
Probability Definitions

CSE 312 Winter 25
Lecture 4

Outline

Today

Two More Rules

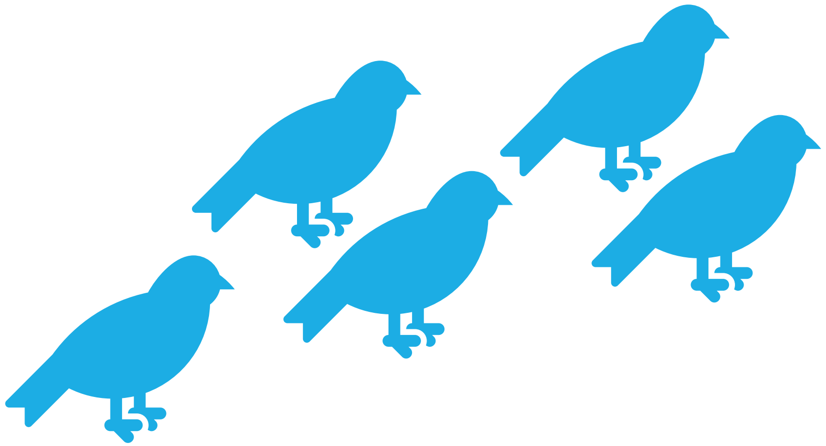
Formal Definitions of Probability

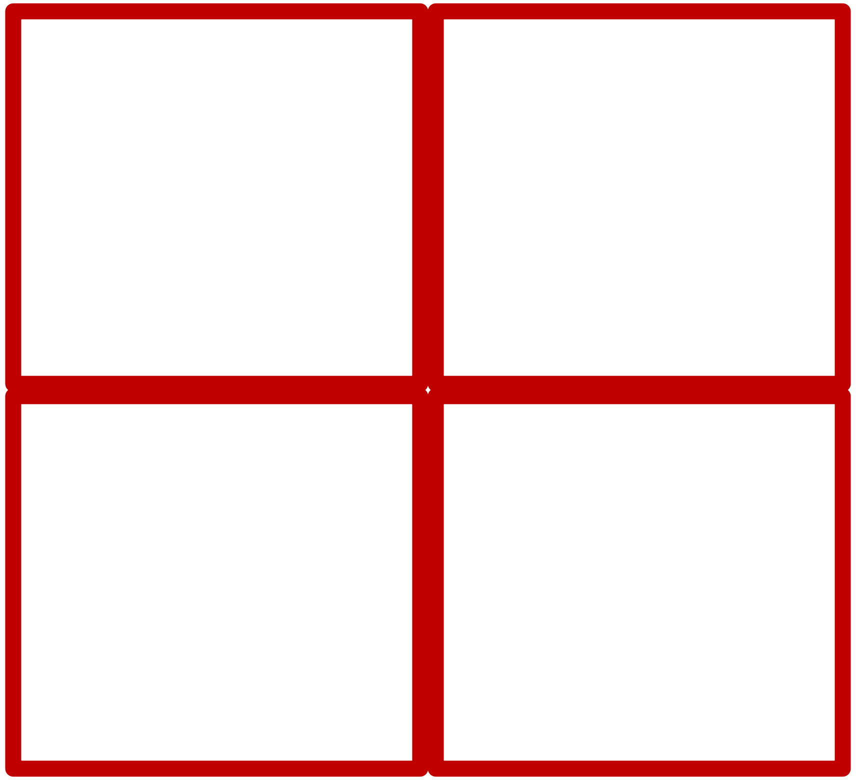
Pigeonhole Principle

Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.



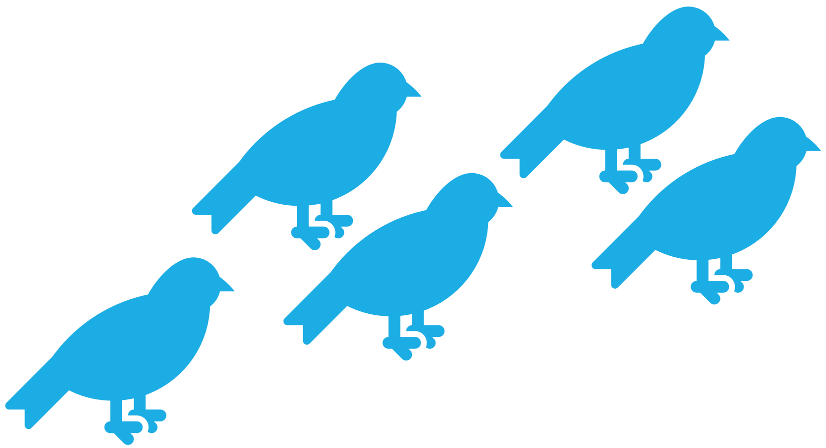


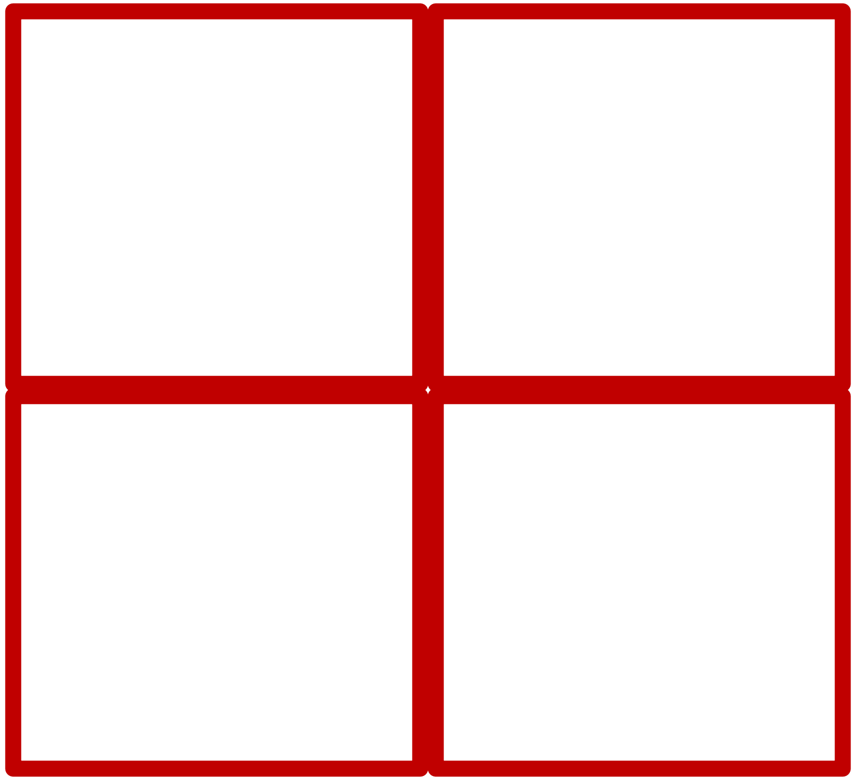
Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.

It might be more than two.





Strong Pigeonhole Principle

If you have n pigeons and k pigeonholes, then there is at least one pigeonhole that has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

$\lceil a \rceil$ is the "ceiling" of a (it means always round up, $\lceil 1.1 \rceil = 2$, $\lceil 1 \rceil = 1$).

An example

If you have to take 10 classes, and have 3 quarters to take them in, then…

Pigeons: The classes to take

Pigeonholes: The quarter

Mapping: Which quarter you take the class in.

Applying the (generalized) pigeonhole principle, there is at least one quarter where you take at least $\left\lceil \frac{10}{3} \right\rceil = 4$ courses.

Practical Tips

When the pigeonhole principle is the right tool, it's usually the first thing you'd think of or the absolute last thing you'd think of.

For **really** tricky ones, we'll warn you in advance that it's the right method (you'll see one in the section handout).

When applying the principle, say:

- What are the pigeons
- What are the pigeonholes
- How do you map from pigeons to pigeonholes

Look for – a set you're trying to divide into groups, where collisions would help you somehow.

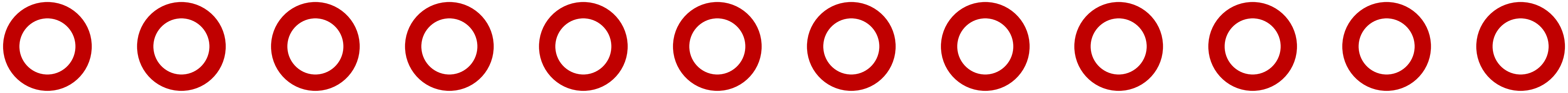
One More Counting Rule

You're going to buy one-dozen donuts (i.e., 12 donuts)

There are chocolate, strawberry, coconut, blueberry, and lemon (i.e. five types)

How many different donut boxes can you buy?

Consider two boxes the same if they contain the same number of every kind of donut (order doesn't matter)



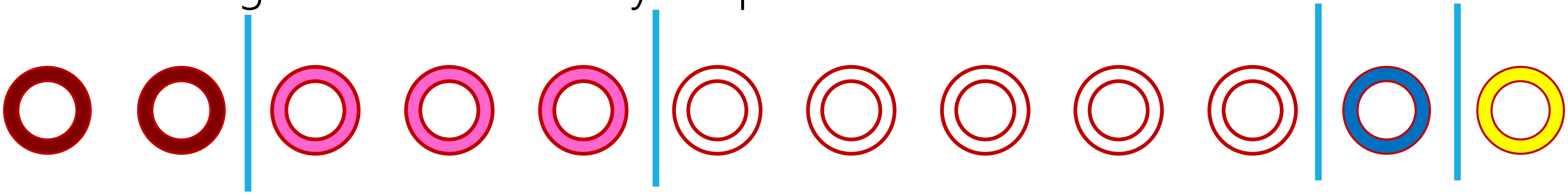
One More Counting Rule

You're going to buy one-dozen donuts (i.e., 12 donuts)

There are chocolate, strawberry, coconut, blueberry, and lemon (i.e. five types)

Put donuts in order by type, then put dividers between the types.

Counting the number of ways to place dividers instead.



Explanation 1

Think of it as a string.

There are $12 + (5 - 1)$ characters.

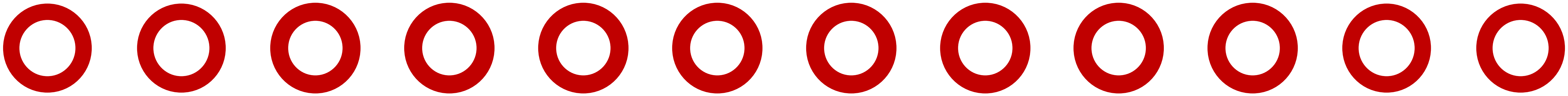
But 12 are the "donut" character (identical) and 4 are the "divider" character (identical).

So?

$$\frac{16!}{12!4!}$$

i.e., $\binom{16}{4}$

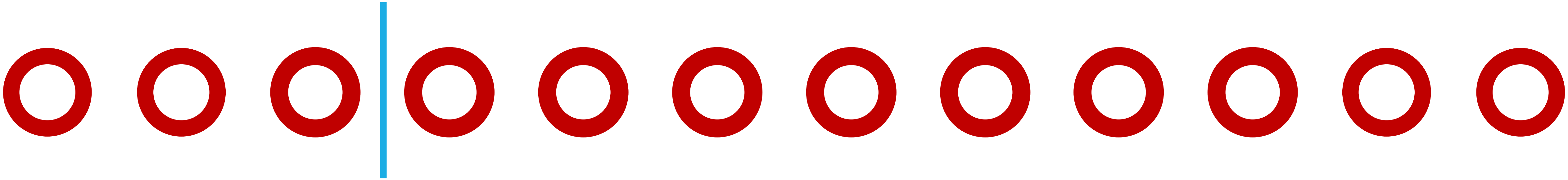
Placing Dividers



Place a divider – how many possible locations are there?

13 – before donut 1, before 2, …, before donut 12, after donut 12.

Placing Dividers



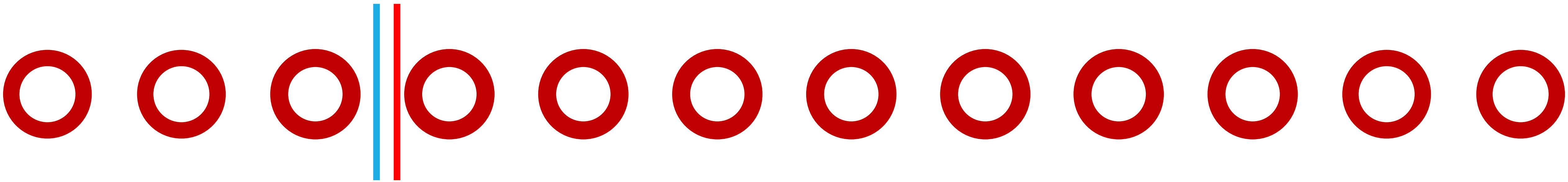
Place a divider – how many possible locations are there?

13 – before donut 1, before 2, …, before donut 12, after donut 12.

Place the second divider, how many possible locations are there?

14 – one of the previous spots was split (“before” and “after” the last divider)

Placing Dividers



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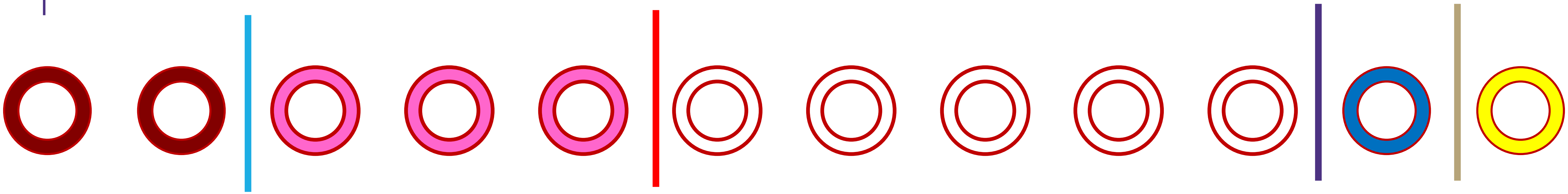
In general, placing divider i has $12 + i$ possible locations.

Wrapping Up



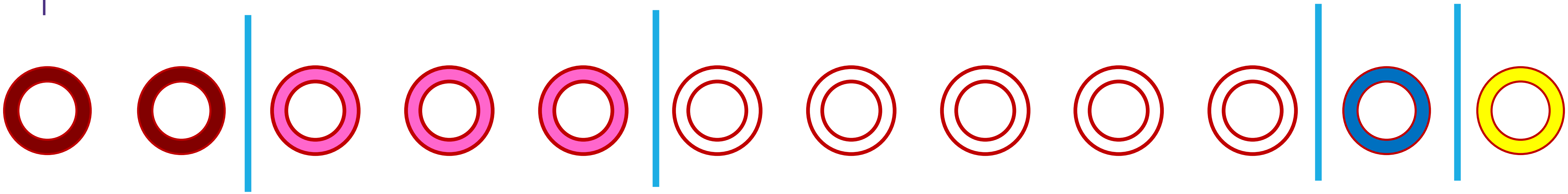
We had 12 donuts, how many dividers do we need?

4 (to divide into 5 groups)

Count so far: $13 \cdot 14 \cdot 15 \cdot 16$

Are we done?

Wrapping Up



Count so far: $13 \cdot 14 \cdot 15 \cdot 16$

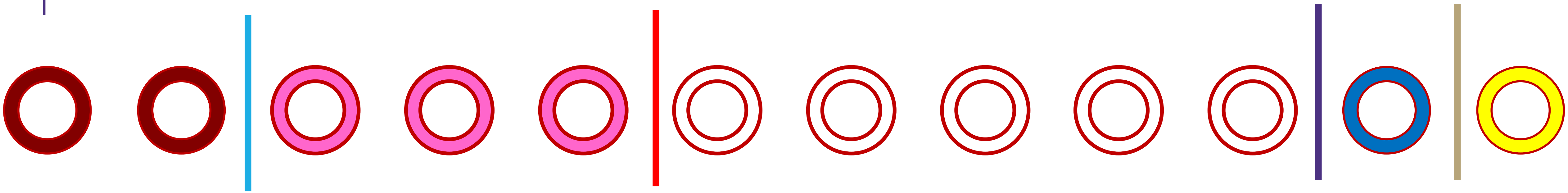
This count treats all dividers as different – they're not! Divide by $4!$.

For n donuts of k types

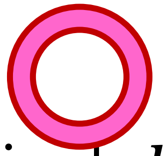
$$\frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}$$

That's a combination! $\binom{n+k-1}{k-1}$

Wrapping Up



$\binom{n+k-1}{k-1}$

We wrote down a "string" consisting of n  and $k-1$

$n+k-1$ characters, n "donuts" are identical, $k-1$ "dividers" are identical, so divide by the rearrangements (like we did for SEATTLE).

In General

To pick n objects from k groups (where order doesn't matter and every element of each group is indistinguishable), use the formula:

$$\binom{n+k-1}{k-1}$$

The counting technique we did is often called "stars and bars"

using a "star" instead of a donut shape, and calling the dividers "bars"

We’ve seen lots of ways to count

Sum rule (split into disjoint sets)

Product rule (use a sequential process)

Combinations (order doesn’t matter)

Permutations (order does matter)

Principle of Inclusion-Exclusion

Complementary Counting

“Stars and Bars” $\binom{n+k-1}{k-1}$

Niche Rules (useful in very specific circumstances)

- Binomial Theorem
- Pigeonhole Principle

Extra Practice

Extra video coming later this week

Practice

How do we know which rule to apply?

With practice you can pick out patterns for which ones might be plausible.

But if as you’re working you realize things are getting out of control, put it aside and try something different.

Cards

A "standard" deck of cards has 52 cards. Each card has a suit

diamonds ♦,

hearts ♥,

clubs ♧,

spades ♠

and a value (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen, King).

A "5-card-hand" is a set of 5 cards

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit.

Cards

How many five-card “flushes” are there? – a flush is a hand of cards all of the same suit.

Way 1: How can I describe a flush? Which suit it is, and which values:

$$\binom{4}{1} \cdot \binom{13}{5}$$

Cards

Way 2: Pretend order matters. The first card can be anything,

After that, you'll have 12 options (the remaining cards of the suit), then 11, …

Then divide by $5!$, since order isn't supposed to matter.

$$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$$

This is the same number as what we got on the last slide!

A Solution with a Problem

You wish to count the number of 5-card hands with at least 3 aces.

There are 4 Aces (and 48 non aces)

$\binom{4}{3} \cdot \binom{49}{2}$

Choose the three aces. Then of the 49 remaining cards (the last ace is allowed as well, because we're allowed to have all 4)

What's wrong with this calculation?

What's the right answer?

Fill out the poll everywhere so Robbie knows how long to explain
Go to pollev.com/robbie

A Solution with a Problem

For a hand, there should be exactly one set of choices in the sequential process that gets us there.

{A♣, A♠, A♦} {A♥, K ♠}

And

{A♣, A♠, A♥}, {A♦, K ♠}

Are two different choices of the process, but they lead to the same hand!

A Solution with a Problem

We could count exactly which hands appear more than once, and how many times each appears and compensate for it.

See the extra slides at the end.

An easier solution is to try again…

The problem was trying to account for the “at least” – come up with disjoint sets and count separately.

$$\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$$

If there are exactly 3 aces, we choose which 3 of the 4, then choose which 2 cards among the 48 non-aces. If all 4 aces appear, then one of the remaining 48 cards finishes the hand. Applying the sum rule completes the calculation.

Takeaway

It's hard to count sets where one of the conditions is "at least X"

You usually need to break those conditions up into disjoint sets and use the sum rule.

Another Problem

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas.

You need to pick at most 2 apples and at least 1 banana. How many sets of fruit can you choose?

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Divide into cases based on number of apples:

0 apples: 1 to 8 bananas possible (8 options)

1 apple: 1 to 7 bananas possible (7 options)

2 apples: 1 to 6 bananas possible (6 options)

21 total (by sum rule)

Another Problem

You have to choose 8 pieces of fruit. There are apples, oranges, and bananas. You need to pick at most 2 apples and at least 1 banana. How many sets of fruit can you choose?

Pick out your first banana. Problem is now to pick 7 fruits (at most 2 apples, allowed to take apples oranges and bananas)

Ignore apple restriction, and subtract off when too many apples:

Ignore restriction: $\binom{7+3-1}{3-1}$

≥ 3 apples, $\binom{4+3-1}{3-1}$ (choose 3 apples first, pick 4 remaining)

Total: $\binom{9}{2} - \binom{6}{2} = 36 - 15 = 21$

Takeaways

For donut-counting style problems with "twists", it sometimes helps to "just throw the first few in the box" to get a problem that is exactly in the donut-counting framework.

When you can do a problem two **very** different ways and get the same answer, you get much more confident in the answer.

Which Tool Do I Use?

Pick k things from universe of n $(n \geq k)$	Repetition is NOT allowed	Repetition IS allowed
Order does NOT matter	Combinations $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$	Stars and Bars $\binom{n+k-1}{n-1}$ Be careful which is n and which is k. This is k donuts from n flavors.
Order does matter	Permutations $P(n,k) = \frac{n!}{(n-k)!}$	Product rule $n \cdot n \cdots n = n^k$

This is **NOT** foolproof! Sometimes you need a twist on the formula; sometimes it's a completely different tool. But a sign where to start.

Fixing The Overcounting

A Solution with a Problem

You wish to count the number of 5-card hands with at least 3 aces.

There are 4 Aces (and 48 non aces)

$\binom{4}{3} \cdot \binom{49}{2}$

Choose the three aces. Then of the 49 remaining cards (the last ace is allowed as well, because we're allowed to have all 4)

What's wrong with this calculation?

When do we overcount?

If there are exactly 4 Aces in the hand, then we count the hand 4 different times (once for each ace as an "extra" one:

{A♧, A♠, A♦}, {A♥, ?}

{A♧, A♠, A♥}, {A♦, ?}

{A♧, A♥, A♦}, {A♠, ?}

{A♥, A♠, A♦}, {A♧, ?}

How much do we overcount?

There are 48 such hands (one for every card that could be "?" on the last slide)

So we've counted $3 \cdot 48$ processes that shouldn't count.

That would give a corrected total of $\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 48$

This is the same number as we got during lecture with our other counting.

Probability

Probability

Probability is a way of **quantifying** our uncertainty.

When more than one outcome is possible,

To have "real-world" examples, we'll need to start with some foundational processes that we're going to assert exist

- We can flip a coin, and each face is equally likely to come up
- We can roll a die, and every number is equally likely to come up
- We can shuffle a deck of cards so that every ordering is equally likely.

Sample Space

Sample Space

A sample space Ω is the set of all possible outcomes of an experiment.

"outcome" is our word for a single element of Ω.

Examples:

For a single coin flip, $\Omega = \{H, T\}$

For a series of two coin flips, $\Omega = \{HH, HT, TH, TT\}$

For rolling a (normal) die: $\Omega = \{1,2,3,4,5,6\}$

Events

Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of Ω)

Examples:

Get at least one head among two coin flips ($E = \{HH, HT, TH\}$)

Get an even number on a die-roll ($E = \{2,4,6\}$).

Examples

I roll a blue 4-sided die and a red 4-sided die.

The table contains the sample space.

The event "the sum of the dice is even" is in gold
The event "the blue die is 1" is in green

	D2=1	D2=2	D2=3	D2=4
D1=1	(1,1)	(1,2)	(1,3)	(1,4)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Probability

Probability

A probability is a number between 0 and 1 describing how likely a particular outcome is.

We'll define a function

$\mathbb{P}: \Omega \to [0,1]$

i.e. $\mathbb{P}$ takes an element of Ω as input and outputs the probability of the outcome.

We'll also use $\Pr[\omega]$, $P(\omega)$ as notation.

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

It depends on what we want to model

If the coin is fair $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$.

But we also might have a biased coin: $\mathbb{P}(H) = .85$, $\mathbb{P}(T) = 0.15$.

Probability Space

Probability Space

A (discrete) probability space is a pair $(\Omega, \mathbb{P})$ where:

Ω is the sample space

$\mathbb{P}: \Omega \to [0, 1]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(x) \geq 0$ for all x
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$
- If $E, F \subseteq \Omega$ and $E \cap F = \emptyset$ then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

Probability Space

Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Is this a valid probability space?

$\mathbb{P}$ takes in elements of Ω and outputs numbers between 0 and 1

$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$.

Measure

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

So what's the probability of seeing a heads?

Seeing heads isn't an element of the sample space!

Define $\mathbb{P}(E) = \sum_{\omega \in \mathrm{E}} \mathbb{P}(\omega)$

Probability Space

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A (discrete) probability space is a pair $(\Omega, \mathbb{P})$ where:

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- If $E, F \subseteq \Omega$ and $E \cap F = \emptyset$ then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

Uniform Probability Space

The most common probability measure is the **uniform** probability measure. In the uniform measure, for every event E

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}.$$

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

A. $\binom{100}{50}/2^{100}$

B. $1/101$

C. $1/2$

D. $1/2^{50}$

E. There is not enough information in this problem.

Mutually Exclusive Events

Two events E, F are mutually exclusive if they cannot happen simultaneously.

In notation, $E \cap F = \emptyset$ (i.e. they're disjoint subsets of the sample space).

For example, if $\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$E_1 =$"the coin came up heads"

$E_2 =$"the coin came up tails"

$E_3 =$"the die showed an even number"

E_1 and E_2 are mutually exclusive.
E_1 and E_3 are not mutually exclusive.

Axioms and Consequences

We wrote down 3 requirements (axioms) on probability measures

- $\mathbb{P}(x) \geq 0$ for all x (non-negativity)
- $\sum_{x\in\Omega} \mathbb{P}(x) \; = 1$ (normalization)
- If E and F are mutually exclusive then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ (countable additivity)

These lead quickly to these three corollaries

- $\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$ (complementation)
- If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$ (monotonicity)
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$ (inclusion-exclusion)

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space?

What is your probability measure $\mathbb{P}$?

What is your event?

What is the probability?

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space? $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

What is your probability measure $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is your event? $\{2,4,6\} \times \{2,4,6\}$

What is the probability? $3^2/6^2$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What if we can't tell the dice apart and always put the dice in increasing order by value.

What is your sample space?

$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(2,5),(2,6)$

$(3,3),(3,4),(3,5),(3,6),(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)\}$

What is your probability measure $\mathbb{P}$?

$\mathbb{P}((x,y)) = 2/36$ if $x \neq y$, $\mathbb{P}(x,x) = 1/36$

What is your event? $\{(2,2),(4,4),(6,6),(2,4),(2,6),(4,6)\}$

What is the probability? $3 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} = \frac{9}{36}$

Takeaways

There is often more than one sample space possible! But one is probably easier than the others.

Finding a sample space that will make the uniform measure correct will usually make finding the probabilities easier to calculate.