Even More Counting

CSE 312 Winter 25
Lecture 3

Announcements

HW 1 is out! Due Wednesday.

Office hours started today! Visit now before others start on the homework.

Also, please start on the homework early.

Announcements

There’s an optional (online) textbook.

It’s linked on the webpage (under the resources tab).

Useful if you want a different perspective.

Occasional differences in notation/vocabulary, but still useful!

Outline

So Far

- Sum and Product Rules
- Combinations (order doesn’t matter) and Permutations (order does matter)
- Introduce ordering and remove it to make calculations easier

This Time

- Some Proofs by counting two ways
- Binomial Theorem
- Principle of Inclusion-Exclusion

Overcounting

How many anagrams are there of SEATTLE

(an anagram is a rearrangement of letters).

It’s not 7! That counts SEATTLE and SEATTLE as different things!

I swapped the Es (or maybe the Ts)

Overcounting

How many anagrams are there of SEATTLE

Pretend the order of the Es (and Ts) relative to each other matter (that SEATTLE and SEATTLE are different)

How many arrangements of SEATTLE? $7!$

How have we overcounted? Es relative to each other and Ts relative to each other $2! \cdot 2!$

Final answer $\frac{7!}{2! \cdot 2!}$

Overcounting

How many anagrams are there of GODOGGY?

Overcounting

How many anagrams are there of GODOGGY?

$\frac{7!}{2!3!}$

One more piece of notation – "multinomial coefficient"

$\binom{7}{2,3}$ is alternate notation for $\frac{7!}{2!3!}$.

In general: $\binom{n}{k_1,k_2,\ldots,k_\ell} = \frac{n!}{k_1!\cdot k_2!\cdots k_\ell!}$

Popular notation among mathematicians.

Combination Facts

Some Facts about combinations

Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Two Proofs of Symmetry

Proof 1: By algebra

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ Definition of Combination

$= \frac{n!}{(n-k)!k!}$ Algebra (commutativity of multiplication)

$= \binom{n}{n-k}$ Definition of Combination

Two Proofs of Symmetry

Wasn’t that a great proof.

Airtight. No disputing it.

Got to say “commutativity of multiplication.”

But…do you know *why*? Can you *feel* why it’s true?

Two Proofs of Symmetry

Suppose you have n people, and need to choose k people to be on your team. We will count the number of possible teams two different ways.

Way 1: We choose the k people to be on the team. Since order doesn't matter (you're on the team or not), there are $\binom{n}{k}$ possible teams.

Way 2: We choose the $n - k$ people to NOT be on the team. Everyone else is on it. Since order again doesn't matter, there are $\binom{n}{n-k}$ possible ways to choose the team.

Since we're counting the same thing, the numbers must be equal.
So $\binom{n}{k} = \binom{n}{n-k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$
\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-1-[k-1])!} + \frac{(n-1)!}{k!(n-1-k)!} && \text{definition of combination} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} && \text{subtraction} \\
&= \frac{[(n-1)!k!(n-k-1)!] + [(n-1)!(k-1)!(n-k)!]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{Find a common denominator} \\
&= \frac{(n-1)!(k-1)!(n-k-1)!\,[k+(n-k)]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{factor out common terms} \\
&= \frac{(n-1)!\,[k+(n-k)]}{k!(n-k)!} && \text{Cancel } (k-1)!(n-k-1)! \\
&= \frac{(n-1)!\cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} && \text{Algebra} \\
&= \binom{n}{k} && \text{Definition of combination}
\end{aligned}
$$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

You and $n-1$ other people are trying out for a k person team. How many possible teams are there?

Way 1: There are n people total, of which we're choosing k (and since it's a team order doesn't matter) $\binom{n}{k}$.

Way 2: There are two types of teams. Those for which you make the team, and those for which you don't.

If you do make the team, then $k-1$ of the other $n-1$ also make it.

If you don't make the team, k of the other $n-1$ also make it.

Overall, by sum rule, $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Since we're computing the same number two different ways, they must be equal. So: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Takeaways

Formulas for factorial, permutations, combinations.

A useful trick for counting is to pretend order matters, then account for the overcounting at the end (by dividing out repetitions)

When trying to prove facts about counting, try to have each side of the equation count the same thing.

Much more fun and much more informative than just churning through algebra.

Three Special Purpose Tools

Binomial Theorem

In high school you probably memorized

$(x + y)^2 = x^2 + 2xy + y^2$

And $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

The Binomial Theorem tells us what happens for every n:

The Binomial Theorem

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Some intuition

The Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Intuition: Every monomial on the right-hand-side has either x or y from each of the terms on the left.

How many copies of $x^i y^{n-i}$ do you get? Well how many ways are there to choose i x's and $n-i$ y's? $\binom{n}{i}$.

Formal proof? Induction!

So What?

Well…if you saw it before, now you have a better understanding now of why it's true.

There are also a few cute applications of the binomial theorem to proving other theorems (usually by plugging in numbers for x and y) – you'll do one on HW1.

For example, set $x = 1$ and $y = 1$ then

$2^n = (1 + 1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}$.

i.e. if you sum up binomial coefficients, you get 2^n. Exercise: reprove this equation (directly) with a combinatorial proof (where have we seen 2^n recently?)

Principle of Inclusion-Exclusion

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:

Exactly 2 'a's OR

Exactly 1 'b' OR

No 'x's

For what A, B, C do we want $|A \cup B \cup C|$?

Why not just use the sum rule?

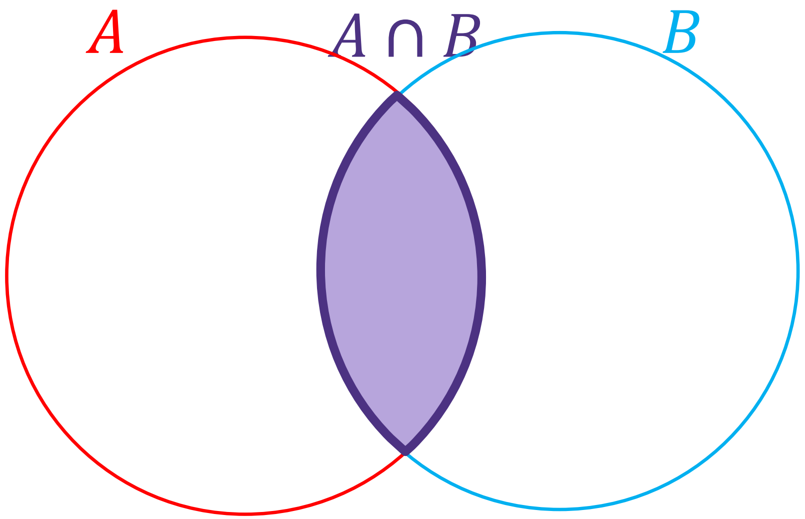
Principle of Inclusion-Exclusion

The sum rule says when A and B are disjoint (no intersection), then $|A \cup B| = |A| + |B|$.

What about when A and B aren't disjoint?

For two sets:

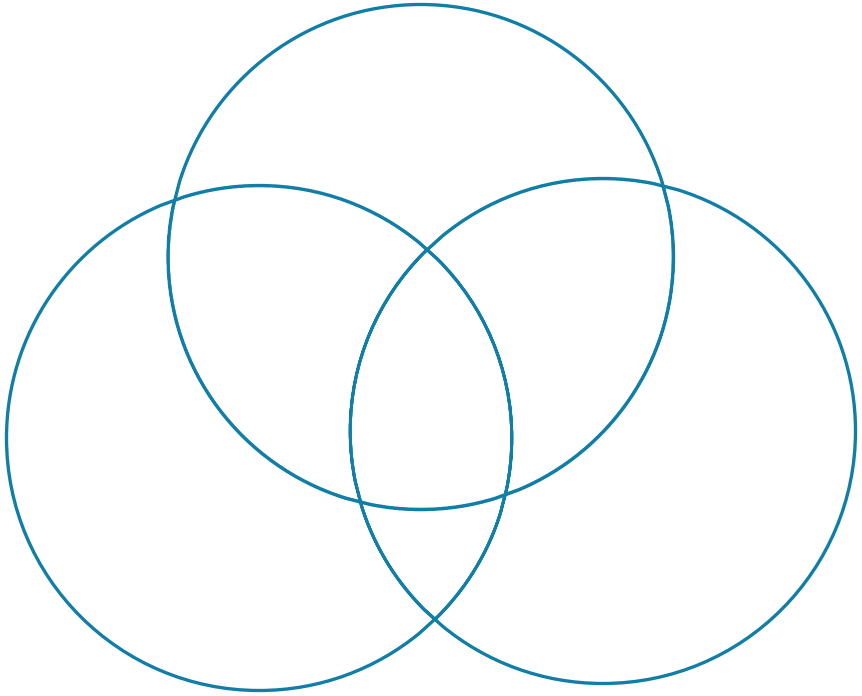
$$|A \cup B| = |A| + |B| - |A \cap B|$$



Principle of Inclusion-Exclusion

For three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:

Exactly 2 ‘a’s OR

Exactly 1 ‘b’ OR

No ‘x’s

For what A, B, C do we want $|A \cup B \cup C|$?

In general:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$|A_1| + |A_2| + \cdots + |A_n|$$

$$-(|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_1 \cap A_n| + |A_2 \cap A_3| + \cdots + |A_{n-1} \cap A_n|)$$

$$+(|A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|)$$

$$- \ \ldots$$

$$+(-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|$$

Add the individual sets, subtract all pairwise intersections, add all three-wise intersections, subtract all four-wise intersections,…, [add/subtract] the n-wise intersection.

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$A = \{$length 5 strings that contain exactly 2 'a's$\}$

$B = \{$length 5 strings that contain exactly 1 'b's$\}$

$C = \{$length 5 strings that contain no 'x's $\}$

$|A| = \binom{5}{2} \cdot 25^3$ (need to choose which "spots" are 'a' and remaining string)

$|B| = \binom{5}{1} \cdot 25^4$

$|C| = 25^5$

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

A = {length 5 strings that contain exactly 2 'a's}

B = {length 5 strings that contain exactly 1 'b's}

C = {length 5 strings that contain no 'x's }

$|A| = \binom{5}{2} \cdot 25^3$

$|B| = \binom{5}{1} \cdot 25^3$

$|C| = 25^5$

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$ (choose 'a' spots, 'b' spot, remaining chars)

$|A \cap C| = \binom{5}{2} \cdot 24^3$ (choose 'a' spots, remaining [non-'x'] chars)

$|B \cap C| = \binom{5}{1} \cdot 24^4$

$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$ (choose 'a' spots, 'b' spot, remaining [non-'x'] chars)

Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$|A| = \binom{5}{2} \cdot 25^3$

$|B| = \binom{5}{1} \cdot 25^4$

$|C| = 25^5$

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$

$|A \cap C| = \binom{5}{2} \cdot 24^3$

$|B \cap C| = \binom{5}{1} \cdot 24^4$

$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$

$$
\begin{aligned}
|A \cup B \cup C| &= \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= \binom{5}{2} \cdot 25^3 + \binom{5}{1} \cdot 25^4 + 25^5 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2 - \binom{5}{2} \cdot 24^3 - \binom{5}{1} \cdot 24^4 + |A \cap B \cap C| \\
&= 11{,}875{,}000 - 1{,}814{,}400 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2 \\
&= 10{,}060{,}600 + 15{,}870 \\
&= 10{,}076{,}470
\end{aligned}
$$

Practical tips

Give yourself clear definitions of A, B, C.

Make a table of all the formulas you need before you start actually calculating.

Calculate “size-by-size” and incorporate into the total.

Basic check: If (in an intermediate step) you ever:

1. Get a negative value
2. Get a value greater than the prior max by adding (after all the single sets)
3. Get a value less than the prior min by subtracting (after all the pairwise intersections)

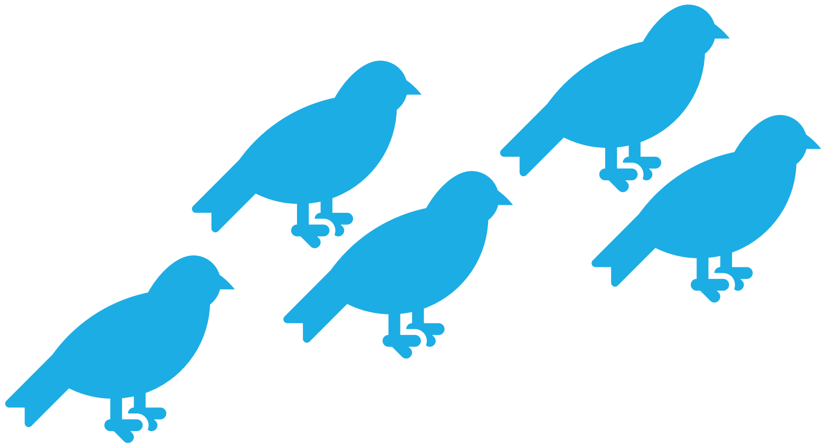
Then something has gone wrong.

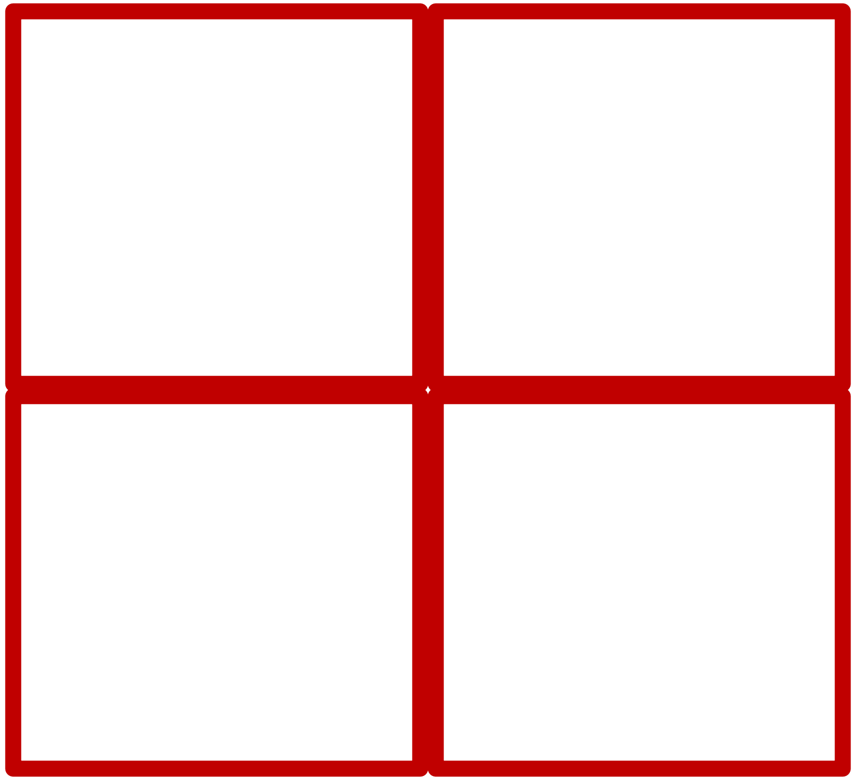
Pigeonhole Principle

Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.



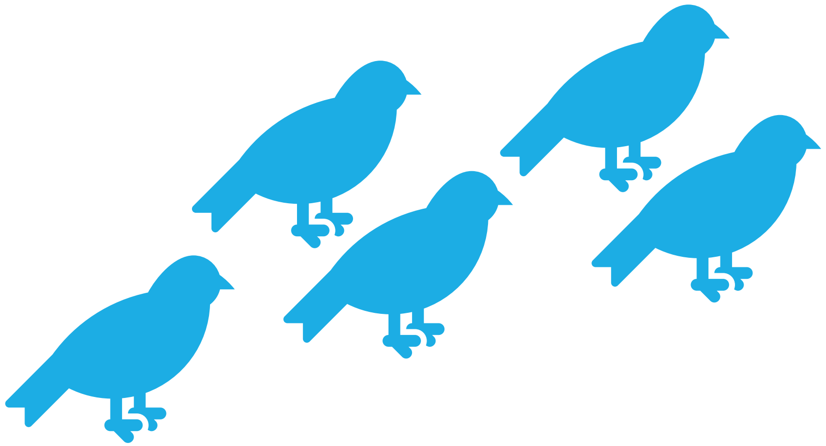


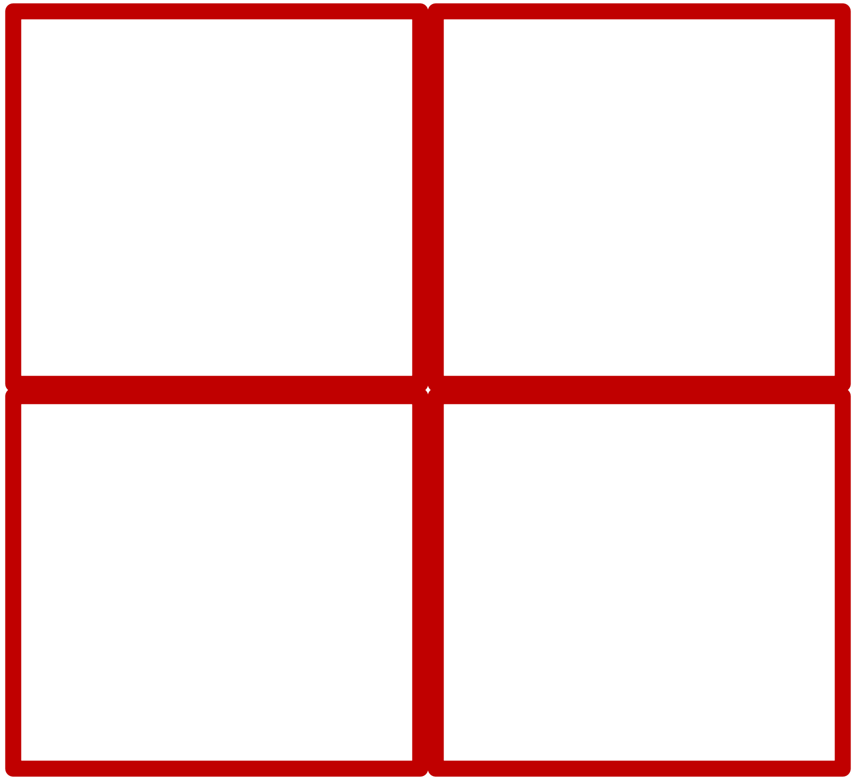
Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.

It might be more than two.





Strong Pigeonhole Principle

If you have n pigeons and k pigeonholes, then there is at least one pigeonhole that has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

$\lceil a \rceil$ is the "ceiling" of a (it means always round up, $\lceil 1.1 \rceil = 2$, $\lceil 1 \rceil = 1$).

An example

If you have to take 10 classes, and have 3 quarters to take them in, then…

Pigeons: The classes to take

Pigeonholes: The quarter

Mapping: Which class you take the quarter in.

Applying the (generalized) pigeonhole principle, there is at least one quarter where you take at least $\left\lceil \frac{10}{3} \right\rceil = 4$ courses.

Practical Tips

When the pigeonhole principle is the right tool, it's usually the first thing you'd think of or the absolute last thing you'd think of.

For **really** tricky ones, we'll warn you in advance that it's the right method (you'll see one in the section handout).

When applying the principle, say:

- What are the pigeons
- What are the pigeonholes
- How do you map from pigeons to pigeonholes

Look for – a set you're trying to divide into groups, where collisions would help you somehow.

One More Formula

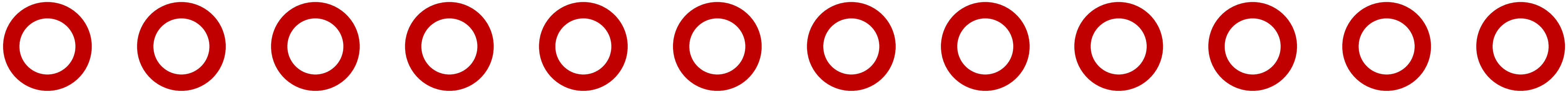
One More Counting Rule

You're going to buy one-dozen donuts (i.e., 12 donuts)

There are chocolate, strawberry, coconut, blueberry, and lemon (i.e. five types)

How many different donut boxes can you buy?

Consider two boxes the same if they contain the same number of every kind of donut (order doesn't matter)



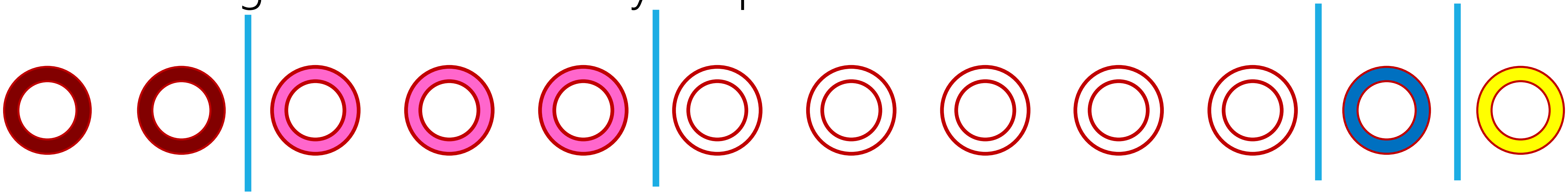
One More Counting Rule

You're going to buy one-dozen donuts (i.e., 12 donuts)

There are chocolate, strawberry, coconut, blueberry, and lemon (i.e. five types)

Put donuts in order by type, then put dividers between the types.

Counting the number of ways to place dividers instead.



Explanation 1

Think of it as a string.

There are $12 + (5 - 1)$ characters.

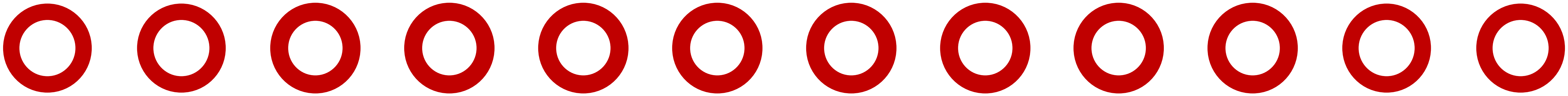
But 12 are the "donut" character (identical) and 4 are the "divider" character (identical).

So?

$$\frac{16!}{12!4!}$$

i.e., $\binom{16}{4}$

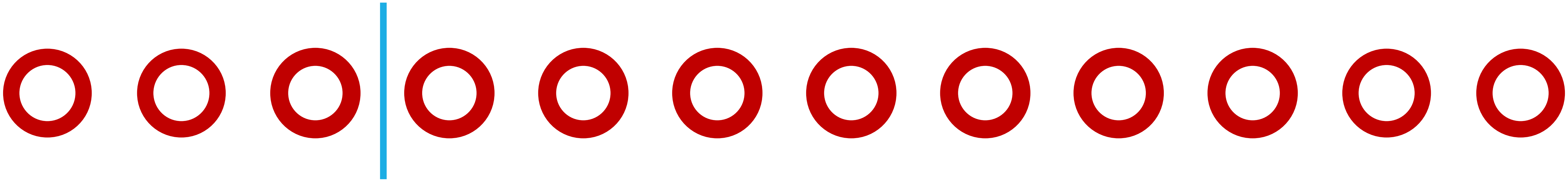
Placing Dividers



Place a divider – how many possible locations are there?

13 – before donut 1, before 2, …, before donut 12, after donut 12.

Placing Dividers



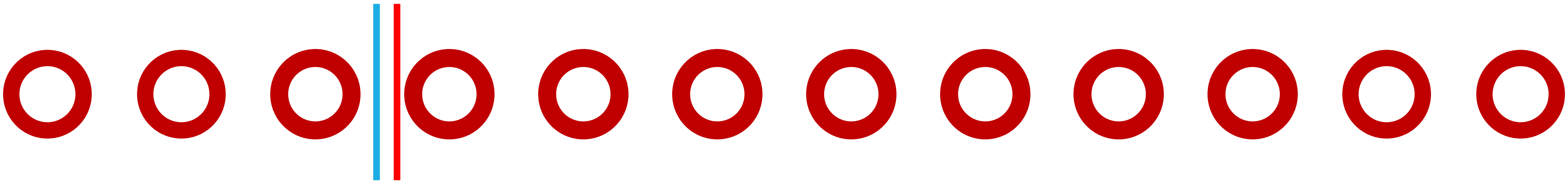
Place a divider – how many possible locations are there?

13 – before donut 1, before 2, …, before donut 12, after donut 12.

Place the second divider, how many possible locations are there?

14 – one of the previous spots was split (“before” and “after” the last divider)

Placing Dividers



Place a divider – how many possible locations are there?

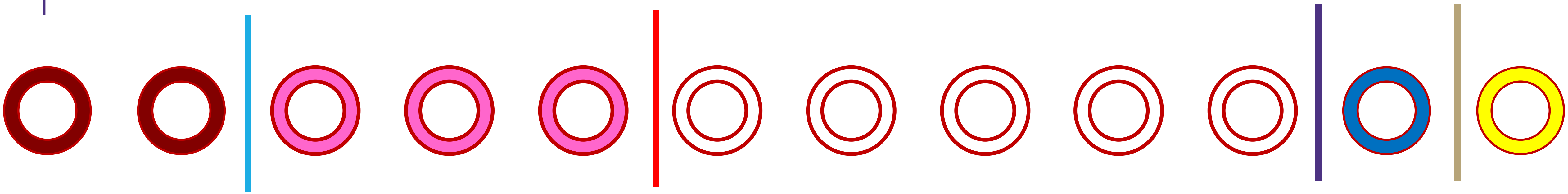
13 – before donut 1, before 2, …, before donut 12, after donut 12.

Place the second divider, how many possible locations are there?

14 – one of the previous spots was split ("before" and "after" the first divider)

In general, placing divider i has $12 + i$ possible locations.

Wrapping Up



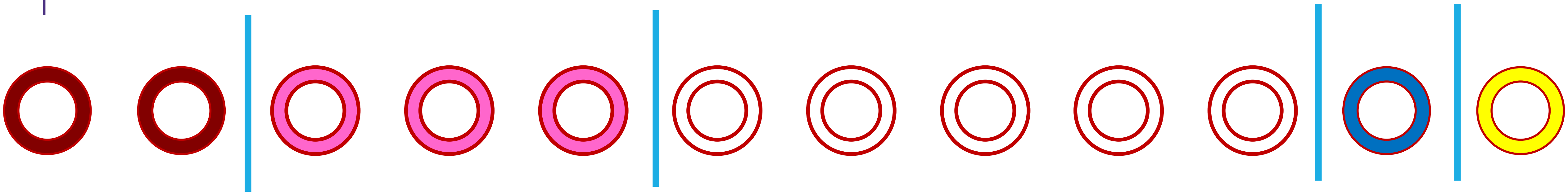
We had 12 donuts, how many dividers do we need?

4 (to divide into 5 groups)

Count so far: $13 \cdot 14 \cdot 15 \cdot 16$

Are we done?

Wrapping Up



Count so far: $13 \cdot 14 \cdot 15 \cdot 16$

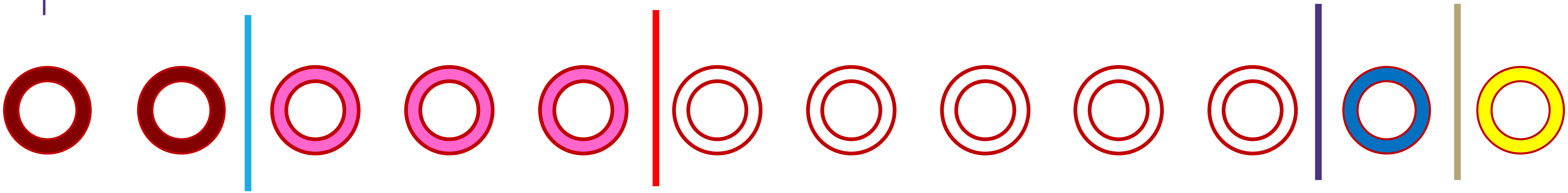
This count treats all dividers as different – they're not! Divide by $4!$.

For n donuts of k types

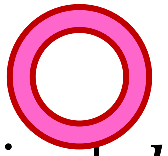
$$\frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}$$

That's a combination! $\binom{n+k-1}{k-1}$

Wrapping Up



$\binom{n+k-1}{k-1}$

We wrote down a "string" consisting of n  and $k-1$ |

$n+k-1$ characters, n "donuts" are identical, $k-1$ "dividers" are identical, so divide by the rearrangements (like we did for SEATTLE).

In General

To pick n objects from k groups (where order doesn't matter and every element of each group is indistinguishable), use the formula:

$$\binom{n+k-1}{k-1}$$

The counting technique we did is often called "stars and bars"

using a "star" instead of a donut shape, and calling the dividers "bars"

We've seen lots of ways to count

Sum rule (split into disjoint sets)

Product rule (use a sequential process)

Combinations (order doesn't matter)

Permutations (order does matter)

Principle of Inclusion-Exclusion

Complementary Counting

"Stars and Bars" $\binom{n+k-1}{k-1}$

Niche Rules (useful in very specific circumstances)

- Binomial Theorem
- Pigeonhole Principle