CSE 312 : Practice Quiz 2 Solutions

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Instructions

- You have twenty-five minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper). You should also get a provided formula sheet (in section it'll be on different colored paper separate from the exam; if you take the exam with DRS it will be the last page of your exam).
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- Since you don't have a calculator, you are generally free to **not** simplify expressions (though you may if you think it will be helpful).
- In general, you should show us the work you used to get to an answer, and explanations will help us reward partial credit, but we do **not** expect explanations at the level we usually require on homeworks.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	Max points
PDFs and CDFs	20
Normals	10
Total	**30**

1. PDFs and CDFs

Let $f_X(x) = \begin{cases} cx^2 & \text{for } 1 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$

(a) What value of c makes the PDF valid?

(b) Compute $\mathbb{P}(1 \le X \le 2)$. For this part and all remaining parts, you may leave c as an unknown constant in the computation.

(c) Find the Expected value of X.

(d) Find the Variance of X.

Solution:

(i) For a pdf we know that the area under the code must be 1. thus

We compute:

$$1 = \int_{-\infty}^{\infty} cx^2 \, dx.$$

Using the bounds:

$$\int_1^5 cx^2 \, dx = \frac{c}{3} x^3 |_1^5 = \frac{124c}{3} = 1 \to c = \frac{3}{124}$$

(ii) Evaluating:

$$\int_1^2 \frac{3}{124} x^2 \, dx = \frac{3}{124 \cdot 3} x^3 |_1^2 = \frac{1}{124} 2^3 - \frac{1}{124} 1^3 = \frac{7}{124} \approx .0564.$$

(iii) Evaluating:

$$E[X] = \int_1^5 x \frac{3}{124} x^2 \, dx = \frac{3}{124 \cdot 4} x^4 |_1^5 = \frac{3}{124 \cdot 4} \cdot 5^4 - \frac{3}{124 \cdot 4} \cdot 1^4 \approx 3.774...$$

(iv) We start by finding $\mathbb{E}[X^2]$.

$$E[X^2] = \int_1^5 x^2 \cdot \frac{3}{124} x^2 \, dx = \frac{3}{124 \cdot 5} x^5 |_1^5 = \frac{3}{124 \cdot 5} 5^5 - \frac{3}{124 \cdot 5} 1^5$$

Now using the formula for variance, and plugging in the answer from the last part:

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{3}{124 \cdot 5} 5^5 - \frac{3}{124 \cdot 5} 1^5 - \left(\frac{3}{124 \cdot 4} \cdot 5^4 - \frac{3}{124 \cdot 4} \cdot 1^4 \right)^2 \approx 0.872.$$

Thus,

(e) Find the PDF for the following CDF. Be sure to include all cases.

Treat n as an unknown positive integer constant.

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y^n & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Solution:

take the derivative of each component of the piecewise

$$f_X(x) = \begin{cases} 0 & \text{otherwise} \\ n \cdot x^{n-1} & 0 \le \text{x} \le 1 \end{cases}$$

2. CLT

Suppose I have a flashlight which requires one battery to operate, and I have 18 identical batteries. I want to go camping for a week ($24 \times 7 = 168$ hours). If the lifetime of a single battery is a random variable distributed as an Exp(0.1)? **Estimate** this probability using the Central Limit Theorem. Do not compute it exactly.

You SHOULD NOT lookup values in the z-table for this problem. Instead, your solution should be an expression with Φ that can be evaluated using the lookup table (i.e. all inputs to Φ are non-negative) and a calculator (i.e. the input to Φ does not need to be simplified).

Solution:

The total lifetime of the battery is

$$X = X_1 + \cdots + X_{18}$$

where each $X_i \sim Exp(0.1)$ has

$$\mathbb{E}[X_i] = \frac{1}{0.1} = 10, \quad \text{and} \quad \text{Var}(X_i) = \frac{1}{0.1^2} = 100.$$

Hence,

$$\mathbb{E}[X] = 180, \quad \text{and} \quad \text{Var}(X) = 1800$$

by linearity of expectation and since variance adds for independent random variables.

By the Central Limit Theorem (CLT),

$$X \approx \mathcal{N}(\mu = 180, \sigma^2 = 1800),$$

so

$$\begin{aligned} \mathbb{P}(X \ge 168) &\approx \mathbb{P}(\mathcal{N}(180, 1800) \ge 168) \\ &= \mathbb{P}\left(Z \ge \frac{168 - 180}{\sqrt{1800}}\right) \\ &= 1 - \Phi\left(\frac{168 - 180}{\sqrt{1800}}\right) \\ &= \Phi\left(\frac{168 - 180}{\sqrt{1800}}\right) \end{aligned}$$

Note that we don't use the continuity correction here because the random variables we are summing are already continuous.

This input to $\Phi()$ is negative, so we'd really need to find $1 - \Phi\left(-1 \cdot \frac{168-180}{\sqrt{1800}}\right)$, though you might not notice that without a caclulator.

Suppose you are using the CLT to approximate the probability $X \le 10$, where X is a binomial random variable with n (the number of trials) equal to 20. What event would you use after doing a continuity correction?

Solution:

X takes on all integer values between 0 and n, so we want the halfway point between integers. 10 (and smaller numbers) are included in the event, but 11 is not, so we use $\mathbb{P}(X \leq 10.5)$.