CSE 312 : Practice Quiz 1 Solutions

Name:	**NetID:** @uw.edu

Instructions

- You have twenty-five minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper). You should also get a provided formula sheet (in section it'll be on different colored paper separate from the exam; if you take the exam with DRS it will be the last page of your exam).
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- Since you don't have a calculator, you are generally free to **not** simplify expressions (though you may if you think it will be helpful).
- In general, you should show us the work you used to get to an answer, and explanations will help us reward partial credit, but we do **not** expect explanations at the level we usually require on homeworks.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	**Max points**
Counting	22
Bayes	17
Multiple Choice	6
Total	**45**

1. (Counting) Clothing [22 points]

Your wardrobe consists of 10 different tops, 10 different jackets, and 10 different pairs of shoes.

(a) How many of the following are there? [3 points each]

- Outfits consisting of one top, one jacket, and one pair of shoes?
- Laundry loads consisting of exactly 10 clothing items?
- Ways in which you can wear your tops, one per day, for the next 5 days without repetition?

Solution:

- 10^3
- $C(30, 10)$
- $P(10, 5)$

(b) How many laundry loads consisting of exactly 10 clothing items that contain at least two tops are there? [3 points] **Solution:**

Let $A =$ set of laundry loads containg at least two tops. We use complementary counting and write, $|A| = |\Omega| - |A^c|$ where $A^c =$ set of laundry loads containg 0 or 1 tops. From question 1 we have $|\Omega| = C(30, 10)$. Using the sum rule we write $|A^c| = |A_0^c| + |A_1^c|$ with the two subsets representing the set of laundry loads containing 0 and 1 tops, respectively. Then, we have:

$$|A^c| = |A_0^c| + |A_1^c| = C(20, 10) + 10 \cdot C(20, 9)$$

Putting altogether we obtain:

$$|A| = C(30, 10) - C(20, 10) - 10 \cdot C(20, 9)$$

(c) Assume that for each item of clothing 5 out of the 10 items are formal. An outfit consists of one top, one jacket, and one pair of shoes. A formal outfit is an outfit where either the jacket and the shoes are formal or where the top and the shoes are formal. How many formal outfits do we have? [5 points]

Solution:

Let $FJ =$ set of formal outfits with a formal jacket and $FT =$ set of formal outfits with a formal top. We have $|FJ| = |FT| = 5^2 \cdot 10$. We can write the set of formal outfits, F, as the union $F = FJ \cup FT$. Then, using inclusion exclusion we obtain:

$$|F| = |FJ \cup FT| = |FJ| + |FT| - |FJ \cap FT| = 2 \cdot 5^2 \cdot 10 - 5^3$$

noting that the intersection $FJ \cap FT$ consist of outfits where all three items are formal.

(d) Packing for a trip, you have room in your suitcase for exactly 16 clothing items. You plan to pack exactly 5 tops, 5 jackets, and 5 pairs of shoes, while the 16th item can be either a top, a jacket, or a pair of shoes. How many different options do you have for the set of clothes that you pack? [5 points]

Solution:

We apply the sum rule to split the set of clothes that we can pack, Ω, into three disjoint subsets $\Omega = T \cap J \cap S$ where T, J, and S represent the sets of clothes that we can pack where the 16th item is the top, jacket, and a pair of shoes respectively. Thus:

$$|\Omega| = |T| + |J| + |S|$$

and

$$|T| = |J| = |S| = C(10, 5) \cdot C(10, 5) \cdot C(10, 6)$$

Thus, the final answer is $|\Omega| = 3 \cdot C(10, 5) \cdot C(10, 5) \cdot C(10, 6)$.

2. (Bayes) Check this box to say you're a human [17 points]

You are designing a CAPTCHA system (a system that checks that users are humans, rather than bots). You know that 80% of the submissions to your system come from bots. If a submission comes from a bot, the bot fails the test 99% of the time; while humans pass the test 75% of the time.

Let H be the event the submission comes from a human, B be the event the submission comes from a bot. Let P be the event the test is passed, and F be the event the test is failed.

(a) Give the notation and fill in the value for "the probability a test is failed, given that the submission came from a human." **be sure to fill in both blanks.** [4 points]

$$\mathbb{P}\left(\underline{\hspace{3cm}}\right) = \underline{\hspace{6cm}}$$

Solution:

$\mathbb{P}(F|H) = .25$

(b) What is the probability a test is failed? [4 points]

Solution:

By Law of Total Probability

$$\mathbb{P}(F) = \mathbb{P}(F|H)\mathbb{P}(H) + \mathbb{P}(F|B)\mathbb{P}(B)$$

Plugging in we get

$$\mathbb{P}(F) = .25 \cdot .2 + .99 \cdot .8$$

(c) You wish to ban the IP addresses of submitters that you think are bots, but you want to be sure they're really bots. What is the probability a submission came from a bot, given that the test failed. [6 points]
You may use b to represent "the correct answer from part b" **Solution:**

By Bayes Rule

$$\mathbb{P}(B|F) = \frac{\mathbb{P}(F|B) \cdot \mathbb{P}(B)}{\mathbb{P}(F)} = \frac{.99 \cdot .8}{b} = \frac{.99 \cdot .8}{.25 \cdot .2 + .99 \cdot .8}$$

(d) Suppose that a human gets frustrated after failing a test once, and so when you show the same human a second test, they have only a 50% change of succeeding. What is the probability of a human failing two consecutive tests? [3 points] **Solution:**

$.25 \cdot .5$

3. Small Questions [6 points]

(a) There are m houses in a suburban neighborhood. Suppose we need to pave a (direct) path between every possible pair of houses. How many paths need to be paved? (Once paved, a path can be used in both directions). [3 points]

◯ m^2 paths
◯ $\frac{m(m-1)}{2}$ paths
◯ $\frac{m}{2}$ paths
◯ $m(m-1)$ paths
◯ None of the above.

Solution:

$\frac{m(m-1)}{2}$ paths. Note that $\binom{m}{2} = \frac{m(m-1)}{2}$.

We can also apply a sequential process of choosing one of m houses and then one of $m-1$ other houses for the end of the path, then dividing by 2 to account for over counting each path (since the path from A to B is the same as the path from B to A).

(b) Your friend attempts to count the number of "two pair" hands. Two pair hands contain:

- Two cards of one value (e.g., two aces or two 8's)
- Two cards of a **different** value
- A fifth card of another different value.

For a standard 52 card deck (13 values, 4 suits), your friend says the number of two pair hands is

$$13\binom{4}{2} \cdot 12\binom{4}{2} \cdot 11\binom{4}{1}.$$

Which best describes their response? [3 points]

◯ It overcounts—you need to divide by $5!$ for all possible reorderings.
◯ It overcounts—you need to divide by $2!$ for reordering the "first pair" compared to the "second pair"
◯ It undercounts—you need to multiply by $5!$ for all possible reorderings.
◯ It undercounts—you need to multiply by $2!$ for reordering the "first pair" compared to the "second pair"

Solution:

It overcounts, you need to divide by $2!$. This sequential process treats (for example) $5\{H, D\}, 4\{H, D\}, 3H$ as different from $4\{H, D\}, 5\{H, D\|3H$, but these produce the same hand, i.e. $\{5H, 5D, 4H, 4D, 3H\}$

We don't divide by $5!$, because a given hand does not correspond to $5!$ choices in the sequential process (only two reorderings).