CSE 312 : Spring 2023 Midterm Exam

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Instructions

- You have ninety minutes to complete this exam.
- You are permitted one piece of 8.5x11 inch paper with handwritten notes (notes are allowed on both sides of the paper).
- You may not use a calculator or any other electronic devices during the exam.
- We will be scanning your exams before grading them. Please write legibly, and avoid writing up to the edge of the paper.
- If you run out of room, you may also use the last page for extra space, but tell us where to find your answer if it's not right below the problem.
- Since you don't have a calculator, you are generally free to **not** simplify expressions (though you may if you think it will be helpful).
- In general, show us the work you used to get to an answer, and explanations will help us reward partial credit, but we do not expect explanations at the level we usually require on homeworks.

Advice

- Writing a few words about where an expression came from is often very helpful for awarding partial credit.
- Remember to take deep breaths.

Question	**Max points**
Short Answer	24
Counting	15
Probability	20
Conditioning	20
Random Variable	25
Grading Morale	1
Total	**105**

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1. [Multiple Choice] A Variance of Topics [24 points]

Each of the questions below has exactly one **correct** answer.
Fully fill in the circle of the best option below.

(a) Suppose $k > 0$ pigeons each go into one of $n > 0$ pigeonholes, numbered 1 through n. (Each pigeonhole can have zero, one, or multiple pigeons in it.) Then...

- ◯ Pigeonhole 1 will always have exactly $\lceil \frac{k}{n} \rceil$ pigeons in it.
- ◯ Pigeonhole 1 will always have **at least** $\lceil \frac{k}{n} \rceil$ pigeons in it (but may not be exactly that many).
- ◯ Neither statement is true.

(b) Suppose events A and B are independent (with $0 < \mathbb{P}(A) < 1$ and $0 < \mathbb{P}(B) < 1$). Then, events A and $\overline{B}$ are independent.

- ◯ Always true
- ◯ Sometimes true
- ◯ Never true

(c) Consider the following potential PMF for a random variable X:

$$p_X(k) = \begin{cases} \frac{1}{4} & k = 3, 6 \\ \frac{3}{4} & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

- ◯ It is not a valid PMF because it needs floors in the ranges for k.
- ◯ It is not a valid PMF because PMFs must sum to 1.
- ◯ It is not a valid PMF because $\lim_{k\to\infty} p_X(k) \neq 1$.
- ◯ It is a valid PMF.

(d) The probability of picking a card that is a queen or a diamond from a standard deck of 52 cards is

$$\frac{4}{52} + \frac{13}{52}$$

.

- ◯ True, this is an application of the sum rule.
- ◯ True, this is an application of inclusion-exclusion.
- ◯ True, this is an application of complementary counting
- ◯ False

Each of the questions below has exactly one **correct** answer.
Fully fill in the circle of the best option below.

(e) True or false:

$$\sum_{n=0}^{75} (-2)^{75-n} \binom{75}{n} = 1$$

- ◯ True
- ◯ False

(f) True or false: the number of ways a people can buy one donut each with b flavor options is a^b.

- ◯ True
- ◯ False

(g) If X is an indicator random variable, then $\mathbb{E}[X]$ is

- ◯ Less than $\mathbb{E}[X^2]$
- ◯ Greater than $\mathbb{E}[X^2]$
- ◯ Equal to $\mathbb{E}[X^2]$
- ◯ Answer depends on the particular indicator random variable.

(h) Suppose you independently roll a fair red die and a fair blue die (both with 6 sides). Let E be the event that the red die shows a 3, F be the event the blue die shows a 5, and G be the event the sum is 8. Which of the following is true?

- ◯ $\mathbb{E}[G] = \mathbb{E}[E] + \mathbb{E}[F]$.
- ◯ F and G are dependent.
- ◯ E and F are dependent.
- ◯ None of the above.

2. [Counting] I Love Bubble Tea [15 points]

(a) Marlene wants to buy 20 bubble teas for a party. There are 4 flavors available - brown sugar, matcha, lychee, and strawberry. If she wants to buy at least 2 of each flavor, and if bubble teas of the same flavor are **indistinguishable**, how many ways are there to select the 20 drinks?

(b) Now, Marlene goes to a different bubble tea shop where each drink is a combination of 4 distinct flavors. In this shop, there are 10 different flavors (two of which are brown sugar and matcha). How many ways can Marlene order 1 drink, such that the drink she ordered contains brown sugar flavor, and/or contains matcha flavor? Consider two orders the same they have the same combination of flavors.

3. [Probability] Society of Cards [20 points]

Suppose we have a deck of cards consisting of the 4 suits ($\clubsuit$, $\diamondsuit$, $\heartsuit$, $\spadesuit$), each with cards ranked from 1 to 13, plus 2 distinguishable Joker cards (Joker A, Joker B). The deck has $(13 \cdot 4) + 2 = 54$ cards total. Suppose you draw a hand of 5 cards from this deck, with each hand equally likely to be drawn.

(a) What is the probability of your hand containing **at least** 1 Joker?

(b) Let J be the event that your hand contains **exactly** 1 Joker. Let S be the event that your hand contains the "1 of Spades" ("1$\spadesuit$") card. Find $\mathbb{P}(J \cup S)$.

(c) What is the probability that your hand is a "Straight", that is, a hand where the cards' ranks form a consecutive sequence?

For example, some **valid** Straights are: {8$\heartsuit$, 9$\clubsuit$, 10$\clubsuit$, 11$\diamondsuit$, 12$\spadesuit$} and {1$\heartsuit$, 2$\heartsuit$, 3$\heartsuit$, 5$\heartsuit$, 4$\heartsuit$}.

Some hands that are **not valid** Straights are: {10$\clubsuit$, 11$\spadesuit$, 12$\heartsuit$, 13$\spadesuit$, 1$\heartsuit$}, {4$\clubsuit$, 5$\diamondsuit$, 6$\spadesuit$, 7$\heartsuit$, 9$\clubsuit$}, {6$\diamondsuit$, 7$\diamondsuit$, 8$\clubsuit$, 8$\diamondsuit$, 9$\diamondsuit$}, {2$\spadesuit$, 4$\heartsuit$, 6$\spadesuit$, 8$\spadesuit$, 10$\heartsuit$}, and {10$\clubsuit$, 11$\spadesuit$, 12$\heartsuit$, 13$\spadesuit$, Joker A}.

Recall that card order in a hand doesn't matter. A straight hand cannot have any Jokers.

4. [Conditioning] Superstition or Reality? [20 points]

Allie and her daughter Sophia love watching Kraken ice hockey games. For some games, one of them will wear the lucky tentacle[1], which affects the Kraken's chances of winning.

- If neither wears the lucky tentacle, the Kraken win with probability $1/2$.
- If Allie wears the lucky tentacle, the Kraken win with probability p (you'll solve for the constant p later).
- If Sophia wears the lucky tentacle, the Kraken win with probability $2p$.
- Note that these are the only possibilities (there is no way for both to wear the one lucky tentacle).

Let A, S, N be the events that Allie, Sophia, or neither wear the lucky tentacle (respectively). Let W be the event that the Kraken win.

(a) Which conditional probability or probability is referred to in the first bullet? Write the appropriate symbols below:

$\mathbb{P}$ (__________)

(b) Sophia has an early bed-time, so she only wears the tentacle $1/5$ of the time. Allie wears it $2/5$ of the time and the remaining $2/5$ neither do. We also know that the Kraken won $\frac{13}{25}$ of their games. Write an expression that will let you solve for p, in terms of only p and numerical values.

(c) Now, solve that formula for p.

[1]The tentacle is the Kraken's replacement for a foam finger. A foam finger is a giant hand-shaped piece of foam to say your team is "#1". You don't need to understand this for the problem

Recall the problem setup from the last page

Allie and her daughter Sophia love watching Kraken hockey games. For some games, one of them will wear the lucky tentacle, which affects the Kraken's chances of winning.

- If neither wears the lucky tentacle, the Kraken win with probability $1/2$.
- If Allie wears the lucky tentacle, the Kraken win with probability p (you'll solve for the constant p later)
- If Sophia wears the lucky tentacle, the Kraken win with probability $2p$.
- Note that these are the only possibilities (there is no way for both to wear the one lucky tentacle).

Let A, S, N be the events that Allie, Sophia, or neither wear the lucky tentacle (respectively). Let W be the event that the Kraken win.

The Kraken won $\frac{13}{25}$ of their games.

(d) If the Kraken won, what is the probability that Sophia was wearing the lucky tentacle?

Write a formula to represent this probability **using only notation** (no numbers nor expressions using p; just probabilities, conditional probabilities, and events) that describe the situation.

(e) Now plug in numbers/expressions in terms of p for each of the parts in the formula you wrote in part (d). **You do not need to simplify**, but may if you think it would help you check your work.

5. [Random Variables] Robbie Can't Sing [25 points]

A group of 5 friends visits a karaoke lounge. The karaoke machine has 20 songs. The friends share the same opinion on every song, so of the available songs: 12 are songs they like, while the remaining 8 are ones they dislike. The group sets the playlist on shuffle, such that every ordering of the 20 songs is equally likely. They decide to take turns singing 4 (consecutive) songs each. A person is **unhappy** if they dislike all 4 songs they sing.

(a) What is the probability that the first person to sing is **unhappy**?

(b) What is the expected number of **unhappy** friends once everyone is done singing?

Now, suppose the 20^{th} song is followed by the 1^{st} song. The shuffle order is the same when looping back to the first song that was played. This means that every song i (where $1 \leq i \leq 20$) is always immediately preceded by one song and immediately followed by another song.

Let X_k be the number of liked songs played immediately before or immediately after the k^{th} disliked song (recall that each song has exactly one "immediately before" and "immediately following" it).

(c) What is Ω_{X_k}, i.e. the support of X_k?

(d) Find the PMF of X_k.

(e) Find the expected number of liked songs played immediately before or immediately after a disliked song using your answer to the previous question.

Recall the problem setup from the last page

(f) A *switch* occurs when a liked song is immediately next to a disliked song. For example, consider the following sequences:

- Disliked Disliked Liked - this sequence contains 1 switch
- Liked Disliked Liked - this sequence contains 2 switches

What is the expected number of times we switch between disliked and liked songs if we play 21 songs (the 20 available songs, followed by the one we started with)? You may use the variable e to represent the answer to part (e) in your answer.

6. Grading Morale [1 point]

Produce a piece of art reflecting your opinion about probability. This might be a poem, a picture of objects in an urn, or anything else.

The TAs look at these to keep morale up when grading.

As long as this page is not empty, you will get the point.

Use this page for extra space if you need it. Be sure to tell us to look here.

Reference Sheet

Theorem: Binomial Theorem

Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Theorem: Principle of Inclusion-Exclusion (PIE)

2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
k events: singles - doubles + triples - quads + ...

Theorem: Pigeonhole Principle

If there are n pigeons we want to put into k holes (where $n > k$), then at least one pigeonhole must contain at least 2 (or to be precise, $\lceil n/k \rceil$) pigeons.

Definition: Key Probability Definitions

The **sample space** is the set Ω of all possible outcomes of an experiment.
An **event** is any subset $E \subseteq \Omega$.
Events E and F are **mutually exclusive** if $E \cap F = \emptyset$.

Definition: Probability space

A *probablity space* is a pair $(\Omega, \mathbb{P})$, where Ω is the sample space
$\mathbb{P} : \Omega \to [0,1]$ is a *probability measure* such that $\sum_{x \in \Omega} \mathbb{P}(x) = 1$.
The probability of an event $E \subseteq \Omega$ is $\mathbb{P}(E) = \sum_{x \in E} \mathbb{P}(x)$.

Definition: Conditional Probability

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

Theorem: Bayes Theorem

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}$$

Definition: Partition

Non-empty events $E_1, \ldots, E_n$ **partition** the sample space Ω if:

- **(Exhaustive)** $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$ (they cover the entire sample space).
- **(Pairwise Mutually Exclusive)** For all $i \neq j$, $E_i \cap E_j = \emptyset$ (none of them overlap)

Theorem: Law of Total Probability (LTP)

If events $E_1, \ldots, E_n$ partition Ω, then for any event F:

$$\mathbb{P}[F] = \sum_{i=1}^{n} \mathbb{P}[F \cap E_i] = \sum_{i=1}^{n} \mathbb{P}[F \mid E_i]\,\mathbb{P}[E_i]$$

Theorem: Bayes Theorem with LTP

Let events $E_1, \ldots, E_n$ partition the sample space Ω, and let F be another event. Then:

$$\mathbb{P}[E_1 \mid F] = \frac{\mathbb{P}[F \mid E_1]\,\mathbb{P}[E_1]}{\sum_{i=1}^{n} \mathbb{P}[F \mid E_i]\,\mathbb{P}[E_i]}$$

Definition: Independence (Events)

A and B are **independent** if any of the following equivalent statements hold:
1. $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$
2. $\mathbb{P}[A \mid B] = \mathbb{P}[A]$
3. $\mathbb{P}[B \mid A] = \mathbb{P}[B]$

Theorem: Chain Rule

Let $A_1, \ldots, A_n$ be events with nonzero probabilities. Then:
$\mathbb{P}[A_1 \cap \cdots \cap A_n] =$
$\mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1]\,\mathbb{P}[A_3 \mid A_1 \cap A_2] \cdots \mathbb{P}[A_n \mid A_1 \cap \cdots \cap A_{n-1}]$

Definition: Mutual Independence (Events)

We say n events $A_1, A_2, \ldots, A_n$ are **(mutually) independent** if, for *any* subset $I \subseteq [n] = \{1, 2, \ldots, n\}$, we have

$$\mathbb{P}\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \mathbb{P}[A_i]$$

This equation is actually representing 2^n equations since there are 2^n subsets of $[n]$.

Definition: Conditional Independence

A and B are **conditionally independent given an event** C if any of the following equivalent statements hold:
1. $\mathbb{P}[A \cap B \mid C] = \mathbb{P}[A \mid C]\,\mathbb{P}[B \mid C]$
2. $\mathbb{P}[A \mid B \cap C] = \mathbb{P}[A \mid C]$
3. $\mathbb{P}[B \mid A \cap C] = \mathbb{P}[B \mid C]$

Definition: Random Variable (RV)

A random variable (RV) X is a numeric function of the outcome $X : \Omega \to \mathbb{R}$. The set of possible values X can take on is its **range/support**, denoted Ω_X.

Definition: Probability Mass Function (PMF)

For a discrete RV X, assigns probabilities to values in its range. That is $p_X : \Omega_X \to [0,1]$ where: $p_X(k) = \mathbb{P}[X = k]$.

Definition: Expectation

The **expectation** of a discrete RV X is: $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$.

Theorem: Linearity of Expectation (LoE)

For any random variables X, Y (possibly dependent):
$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

Theorem: Law of the Unconscious Statistician (LOTUS)

For a discrete RV X and function g, $\mathbb{E}[g(X)] = \sum_{b \in \Omega_X} g(b) \cdot p_X(b)$.

Definition: Variance

$\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$.

Theorem: Property of Variance

$\text{Var}(aX + b) = a^2 \text{Var}(X)$.

Definition: Independence (Random Variables)

Random variables X and Y are **independent** if for *all* $x \in \Omega_X$ and all $y \in \Omega_Y$:
$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x] \cdot \mathbb{P}[Y = y].$$

Theorem: Variance Adds for Independent RVs

If X, Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.