Section 5: Solutions

Review of Main Concepts

- **Independence**: Random variable X and event E are independent iff

$$\forall x, \quad \mathbb{P}(X = x \cap E) = \mathbb{P}(X = x)\mathbb{P}(E)$$

  Random variables X and Y are independent iff

$$\forall x \forall y, \quad \mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

  In this case, we have $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ (the converse is not necessarily true).

- **i.i.d. (independent and identically distributed)**: Random variables $X_1, \ldots, X_n$ are i.i.d. (or iid) iff they are mutually independent and have the same probability mass function.

- **Independence of functions of a r.v.**: If X and Y are independent and $g(\cdot), h(\cdot)$ are functions mapping real numbers to real numbers, then $g(X)$ and $h(Y)$ are independent. (See if you can prove this!)

- **Variance of Independent Variables**: If X is independent of Y, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$. This depends on independence, whereas linearity of expectation always holds. Note that this combined with the above shows that $\forall a, b, c \in \mathbb{R}$ and if X is independent of Y, $Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$.

- Review: Zoo of Discrete Random Variables

  (a) **Uniform**: $X \sim \mathsf{Uniform}(a, b)$ ($\mathsf{Unif}(a, b)$ for short), for integers $a \leq b$, iff X has the following probability mass function:

$$p_X(k) = \frac{1}{b - a + 1}, \quad k = a, a + 1, \ldots, b$$

  $\mathbb{E}[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)(b-a+2)}{12}$. This represents each integer from $[a, b]$ to be equally likely. For example, a single roll of a fair die is $\mathsf{Uniform}(1, 6)$.

  (b) **Bernoulli (or indicator)**: $X \sim \mathsf{Bernoulli}(p)$ ($\mathsf{Ber}(p)$ for short) iff X has the following probability mass function:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1 - p, & k = 0 \end{cases}$$

  $\mathbb{E}[X] = p$ and $Var(X) = p(1-p)$. An example of a Bernoulli r.v. is one flip of a coin with $\mathbb{P}(\text{head}) = p$.

  (c) **Binomial**: $X \sim \mathsf{Binomial}(n, p)$ ($\mathsf{Bin}(n, p)$ for short) iff X is the sum of n iid $\mathsf{Bernoulli}(p)$ random variables. X has probability mass function

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

  $\mathbb{E}[X] = np$ and $Var(X) = np(1-p)$. An example of a Binomial r.v. is the number of heads in n independent flips of a coin with $\mathbb{P}(\text{head}) = p$. Note that $\mathsf{Bin}(1, p) \equiv \mathsf{Ber}(p)$. As $n \to \infty$ and $p \to 0$, with $np = \lambda$, then $\mathsf{Bin}(n, p) \to \mathsf{Poi}(\lambda)$. If $X_1, \ldots, X_n$ are independent Binomial r.v.'s, where $X_i \sim \mathsf{Bin}(N_i, p)$, then $X = X_1 + \ldots + X_n \sim \mathsf{Bin}(N_1 + \ldots + N_n, p)$.

  (d) **Geometric:** $X \sim \mathsf{Geometric}(p)$ ($\mathsf{Geo}(p)$ for short) iff X has the following probability mass function:

$$p_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots$$

  $\mathbb{E}[X] = \frac{1}{p}$ and $Var(X) = \frac{1-p}{p^2}$. An example of a Geometric r.v. is the number of independent coin flips up to and including the first head, where $\mathbb{P}(\text{head}) = p$.

(e) **Poisson**: $X \sim \mathsf{Poisson}(\lambda)$ ($\mathsf{Poi}(\lambda)$ for short) iff X has the following probability mass function:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

$\mathbb{E}[X] = \lambda$ and $Var(X) = \lambda$. An example of a Poisson r.v. is the number of people born during a particular minute, where λ is the average birth rate per minute. If $X_1, \ldots, X_n$ are independent Poisson r.v.'s, where $X_i \sim \mathsf{Poi}(\lambda_i)$, then $X = X_1 + \ldots + X_n \sim \mathsf{Poi}(\lambda_1 + \ldots + \lambda_n)$.

(f) **Negative Binomial**: $X \sim \mathsf{NegativeBinonial}(r, p)$ ($\mathsf{NegBin}(r, p)$ for short) iff X is the sum of r iid $\mathsf{Geometric}(p)$ random variables. X has probability mass function

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots$$

$\mathbb{E}[X] = \frac{r}{p}$ and $Var(X) = \frac{r(1-p)}{p^2}$. An example of a Negative Binomial r.v. is the number of independent coin flips up to and including the r^{th} head, where $\mathbb{P}(\text{head}) = p$. If $X_1, \ldots, X_n$ are independent Negative Binomial r.v.'s, where $X_i \sim \mathsf{NegBin}(r_i, p)$, then $X = X_1 + \ldots + X_n \sim \mathsf{NegBin}(r_1 + \ldots + r_n, p)$.

(g) **Hypergeometric**: $X \sim \mathsf{HyperGeometric}(N, K, n)$ ($\mathsf{HypGeo}(N, K, n)$ for short) iff X has the following probability mass function:

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad k = \max\{0, n+K-N\}, \ldots, \min\{K, n\}$$

$\mathbb{E}[X] = n\frac{K}{N}$. This represents the number of successes drawn, when n items are drawn from a bag with N items (K of which are successes, and $N - K$ failures) without replacement. If we did this with replacement, then this scenario would be represented as Bin $\left(n, \frac{K}{N}\right)$.

1. Content Review Questions

(a) True or false: $Var(A + B) = Var(A) + Var(B)$ **Solution:**

False. This property only holds if A and B are independent.

(b) What is $Var(3A + 4)$?

- ◯ $3Var(A) + 4$
- ◯ $3Var(A)$
- ◯ $9Var(A)$
- ◯ $Var(A)$

Solution:

$9Var(A)$ by the property of variance

(c) True or false: $\mathbb{E}[A + B] = \mathbb{E}[A] + \mathbb{E}[B]$ **Solution:**

True. This is by the linearity of expectation. A and B do not have to be independent.

(d) What is $\mathbb{E}[3A + 4]$?

- ◯ $3\mathbb{E}[A] + 4$

◯ $3\mathbb{E}[A]$

◯ $9\mathbb{E}[A]$

◯ $\mathbb{E}[A]$

Solution:

$3\mathbb{E}[A] + 4$ by the linearity of expectation.

2. Pond Fishing

Suppose I am fishing in a pond with B blue fish, R red fish, and G green fish, where $B + R + G = N$. Each fish is equally likely to be caught. For each of the following scenarios, identify the most appropriate distribution (with parameter(s)):

(a) How many of the next 10 fish I catch are blue, if I catch and release

◯
$$\text{Bin}\left(10, \frac{B}{N}\right)$$

◯
$$\text{Ber}\left(\frac{B}{N}\right)$$

◯
$$\text{Bin}\left(1, \frac{B}{N}\right)$$

Solution:

Since this is the same as saying how many of my next 10 trials (fish) are a success (are blue), this is a binomial distribution. Specifically, since we are doing catch and release, the probability of a given fish being blue is $\frac{B}{N}$ and each trial is independent. Thus:

$$\text{Bin}\left(10, \frac{B}{N}\right)$$

(b) How many fish I had to catch until my first green fish, if I catch and release

◯
$$\text{Ber}\left(\frac{G}{N}\right)$$

◯
$$\text{Bin}\left(1, \frac{G}{N}\right)$$

◯
$$\text{Geo}\left(\frac{G}{N}\right)$$

Solution:

Once again, each catch is independent, so this is asking how many trials until we see a success, hence it is a geometric distribution:

$$\text{Geo}\left(\frac{G}{N}\right)$$

(c) How many red fish I catch in the next five minutes, if I catch on average r red fish per minute

- ◯ $$\text{Poi}(5R)$$
- ◯ $$\text{Bin}\left(5, \frac{R}{N}\right)$$
- ◯ $$\text{Poi}(5r)$$

Solution:

This is asking for the number of occurrences of event given an average rate, which is the definition of the Poisson distribution. Since we're looking for events in the next 5 minutes, that is our time unit, so we have to adjust the average rate to match (r per minute becomes $5r$ per 5 minutes).

$$\text{Poi}(5r)$$

(d) Whether or not my next fish is blue

- ◯ $$\text{Poi}(5B)$$
- ◯ $$\text{Bin}\left(1, \frac{R}{N}\right)$$
- ◯ $$\text{Ber}\left(\frac{B}{N}\right)$$

Solution:

This is the same as the binomial case, but it's only one trial, so it is necessarily Bernoulli.

$$\text{Ber}\left(\frac{B}{N}\right)$$

(e) How many of the next 10 fish I catch are blue, if I do not release the fish back to the pond after each catch
Solution:

Hypergeometric's definition is the number of successes in n draws (without replacement) from N items that contain K successes in total. In this case, we have 10 draws (without replacement because we do not catch and release), and out of the N fish, B are blue (a success).

$$\mathsf{HypGeo}(N, B, 10)$$

(f) How many fish I have to catch until I catch three red fish, if I catch and release **Solution:**

Negative binomial models the number of trials with probability of success p, until you get r successes. In this case, as before, our trials are caught fish (with replacement this time) and our success is if the fish are red, which happens with probability $\frac{R}{N}$.

$$\mathsf{NegBin}\left(3, \frac{R}{N}\right)$$

3. Balls in Bins

Note: this problem also appeared on the section 4 handout.
Let X be the number of bins that remain empty when m balls are distributed into n bins randomly and independently. For each ball, each bin has an equal probability of being chosen. (Notice that two bins being empty are not independent events: if one bin is empty, that decreases the probability that the second bin will also be empty. This is particularly obvious when $n = 2$ and $m > 0$.) Find $\mathbb{E}[X]$. **Solution:**

For $i \in [n]$, let X_i be 1 if bin i is empty, and 0 otherwise. Then, $X = \sum_{i=1}^{n} X_i$. We first compute the expectation of an individual X_i:

$$\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \left(\frac{n-1}{n}\right)^m .$$

Indeed, we are assuming multiple balls can go in the same bin. As such, when computing $P(X_i = 1)$, given that bin i is empty, we remove it from the pool of possible bins to pick from, leaving us with $n-1$ bins out of a total of n bins in which we can place balls. Since we are distributing m balls over the n bins, the event that bin i remains empty occurs with probability $\left(\frac{n-1}{n}\right)^m$. Hence, by linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \left(\frac{n-1}{n}\right)^m = n \cdot \left(\frac{n-1}{n}\right)^m .$$

4. 3-sided Die

Note: a variation of this problem also appeared on the section 4 handout. Let the random variable X be the sum of two independent rolls of a fair 3-sided die. (If you are having trouble imagining what that looks like, you can use a 6-sided die and change the numbers on 3 of its faces.)

(a) What is the probability mass function of X?

Solution:

First let us define the range of X. A three sided-die can take on values $1, 2, 3$. Since X is the sum of two rolls, the range of X is $\Omega_X = \{2, 3, 4, 5, 6\}$.

We can then define the pmf of X. To that end, we must define two random variables R_1, R_2 with R_1 being the roll of the first die, and R_2 being the roll of the second die. Then, $X = R_1 + R_2$. Note that

$\Omega_{R1} = \Omega_{R2} = \{1, 2, 3\}$. With that in mind we can find the pmf of X:

$$\begin{aligned} p_X(k) = \mathbb{P}(X = k) &= \sum_{i \in \Omega_{R1}} \mathbb{P}(R_1 = i, R_2 = k - i) \\ &= \sum_{i \in \Omega_{R1}} \mathbb{P}(R_1 = i) \cdot \mathbb{P}(R_2 = k - i) \qquad \text{(By independence of the rolls)} \\ &= \sum_{i \in \Omega_{R1}} \frac{1}{3} \cdot p_{R2}(k - i) \\ &= \frac{1}{3} \left(p_{R2}(k-1) + p_{R2}(k-2) + p_{R2}(k-3) \right) \end{aligned}$$

At this point, we can evaluate the pmf of X for each value in the range of X, noting that $p_{R2}(k - i) = 0$ if $k - i \notin \Omega_{R2}$, $1/3$ otherwise. We get:

$$p_X(k) = \begin{cases} 1/9 & k = 2 \\ 2/9 & k = 3 \\ 3/9 & k = 4 \\ 2/9 & k = 5 \\ 1/9 & k = 6 \\ 0 & \text{otherwise} \end{cases}$$

One could also list out the possible values of the first two rolls and use a table to find the marginal pmf of X by summing up the entries of each row for each $k \in \Omega_X$.

(b) Find $\mathbb{E}[X]$.

Solution:

There are two ways to find the expected value of X. We could apply the *definition of expectation* using the PMF found in part (a). This gives us

$$\mathbb{E}[X] = \sum_{k=2}^{6} k p_X(k) = 2 \cdot \frac{1}{9} + 3 \cdot \frac{2}{9} + 4 \cdot \frac{3}{9} + 5 \cdot \frac{2}{9} + 6 \cdot \frac{1}{9} = \boxed{4}$$

Alternatively, we can use *linearity of expectation* here. Let R_1 be the roll of the first die, and R_2 the roll of the second. Then, $X = R_1 + R_2$.
By linearity of expectation, we get:

$$\mathbb{E}[X] = \mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$$

We compute:

$$\mathbb{E}[R_1] = \sum_{i \in \Omega_{R1}} i \cdot \mathbb{P}(R_1 = i) = \sum_{i \in \Omega_{R1}} i \cdot \frac{1}{3} = \frac{1}{3}(1 + 2 + 3) = 2$$

Similarly, $E[R_2] = 2$, since the rolls are independent.

Plugging into our expression for the expectation of X gives us:

$$E[X] = 2 + 2 = \boxed{4}$$

(c) What is $Var(X)$?

Solution:

We know from the definition of variance that

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

We can compute the $\mathbb{E}[X^2]$ term as follows:

$$\mathbb{E}[X^2] = \sum_{x=2}^{6} x^2 p_X(x) = \frac{2^2 \cdot 1 + 3^2 \cdot 2 + 4^2 \cdot 3 + 5^2 \cdot 2 + 6^2 \cdot 1}{9} = \frac{52}{3}$$

Plugging this into our variance equation gives us

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{52}{3} - 4^2 = \boxed{\frac{4}{3}}$$

5. Best Coach Ever!!

You are a hardworking boxer. Your coach tells you that the probability of your winning a boxing match is 0.2 independently of every other match.

(a) How many matches do you expect to fight until you win 10 times and what kind of random variable is this?
Solution:

The number of matches you have to fight until you win 10 times can be modeled by $\sum_{i=1}^{10} X_i$ where $X_i \sim \mathsf{Geometric}(0.2)$ is the number of matches you have to fight to go from $i-1$ wins to i wins, including the match that gets you your i^{th} win, where every match has a 0.2 probability of success. Recall $\mathbb{E}[X_i] = \frac{1}{0.2} = 5$. $\mathbb{E}[\sum_{i=1}^{10} X_i] = \sum_{i=1}^{10} \mathbb{E}[X_i] = \sum_{i}^{10} \frac{1}{0.2} = 10 \cdot 5 = 50$.

(b) You only get to play 12 matches every year. To win a spot in the Annual Boxing Championship, a boxer needs to win at least 10 matches in a year. What is the probability that you will go to the Championship this year and what kind of random variable is the number of matches you win out of the 12? **Solution:**

You can go to the championship if you win more than or equal to 10 times this year. Let Y be the number of matches you win out of the 12 matches. Note that $Y \sim \mathsf{Binomial}(12, 0.2)$. Since the max number you can win is 12 (there are 12 matches), we are looking for $P(10 \leq Y \leq 12)$. Thus, since Y is discrete, we are interested in

$$\mathbb{P}(Y = 10) + \mathbb{P}(Y = 11) + \mathbb{P}(Y = 12) = \sum_{i=10}^{12} \binom{12}{i} 0.2^i (1 - 0.2)^{12-i}$$

(c) Let p be your answer to part (b). How many times can you expect to go to the Championship in your 20 year career? **Solution:**

The number of times you go to the championship can be modeled by $Y \sim \mathsf{Binomial}(20, p)$. So, $E[Y] = 20 \cdot p$.

6. Variance of a Product

Let X, Y, Z be independent random variables with means μ_X, μ_Y, μ_Z and variances $\sigma_X^2, \sigma_Y^2, \sigma_Z^2$, respectively. Find $Var(XY - Z)$. **Solution:**

First notice that $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \implies \mathbb{E}[X^2] = Var(X) + \mathbb{E}[X]^2 = \sigma_X^2 + \mu_X^2$, and same for Y.

$$Var(XY) = \mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2 \text{(by theorem in class)}$$
$$= \mathbb{E}[X^2]\mathbb{E}[Y^2] - (\mathbb{E}[X]\mathbb{E}[Y])^2 \text{(by independence)}$$
$$= \mathbb{E}[X^2]\mathbb{E}[Y^2] - \mathbb{E}[X]^2\mathbb{E}[Y]^2$$
$$= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2$$

By independence,

$$Var(XY - Z) = Var(XY) + Var(-Z) = Var(XY) + Var(Z)$$
$$= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 + \sigma_Z^2$$

7. True or False?

Identify the following statements as true or false (true means always true). Justify your answer.

(a) For any random variable X, we have $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$. **Solution:**

True, since $0 \leq Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$, since the squaring necessitates the result is non-negative.

(b) Let X, Y be random variables. Then, X and Y are independent if and only if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. **Solution:**

False. The forward implication is true, but the reverse is not. For example, if X is the discrete uniform random variable on the set $\{-1, 0, 1\}$ such that $P(X = -1) = P(X = 0) = P(X = 1) = \frac{1}{3}$, and $Y = X^2$, we have $\mathbb{E}[X] = 0$, so $\mathbb{E}[X]\mathbb{E}[Y] = 0$. However, since $X = X^3$, $\mathbb{E}[XY] = \mathbb{E}[XX^2] = \mathbb{E}[X^3] = \mathbb{E}[X] = 0$, we have that $\mathbb{E}[X]\mathbb{E}[Y] = 0 = \mathbb{E}[XY]$. However, X and Y are not independent; indeed, $\mathbb{P}(Y = 0|X = 0) = 1 \neq \frac{1}{3} = \mathbb{P}(Y = 0)$.

(c) Let $X \sim \mathsf{Binomial}(n, p)$ and $Y \sim \mathsf{Binomial}(m, p)$ be independent. Then, $X + Y \sim \mathsf{Binomial}(n + m, p)$. **Solution:**

True. X is the sum of n independent Bernoulli trials, and Y is the sum of m. So $X + Y$ is the sum of $n + m$ independent Bernoulli trials, so $X + Y \sim \mathsf{Binomial}(n + m, p)$.

(d) Let $X_1, ..., X_{n+1}$ be independent $\mathsf{Bernoulli}(p)$ random variables. Then, $\mathbb{E}[\sum_{i=1}^{n} X_i X_{i+1}] = np^2$. **Solution:**

True. Notice that $X_i X_{i+1}$ is also Bernoulli (only takes on 0 and 1), but is 1 iff both are 1, so $X_i X_{i+1} \sim \mathsf{Bernoulli}(p^2)$. The statement holds by linearity, since $\mathbb{E}[X_i X_{i+1}] = p^2$.

(e) Let $X_1, ..., X_{n+1}$ be independent $\mathsf{Bernoulli}(p)$ random variables. Then, $Y = \sum_{i=1}^{n} X_i X_{i+1} \sim \mathsf{Binomial}(n, p^2)$. **Solution:**

False. They are all Bernoulli p^2 as determined in the previous part, but they are not independent. Indeed, $\mathbb{P}(X_1X_2 = 1|X_2X_3 = 1) = \mathbb{P}(X_1 = 1) = p \neq p^2 = \mathbb{P}(X_1X_2 = 1)$.

(f) If $X \sim$ Bernoulli(p), then $nX \sim$ Binomial(n, p). **Solution:**

False. The range of X is $\{0, 1\}$, so the range of nX is $\{0, n\}$. nX cannot be Bin(n, p), otherwise its range would be $\{0, 1, ..., n\}$.

(g) If $X \sim$ Binomial(n, p), then $\frac{X}{n} \sim$ Bernoulli(p). **Solution:**

False. Again, the range of X is $\{0, 1, ..., n\}$, so the range of $\frac{X}{n}$ is $\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$. Hence it cannot be Ber$(p)$, otherwise its range would be $\{0, 1\}$.

(h) For any two independent random variables X, Y, we have $Var(X - Y) = Var(X) - Var(Y)$. **Solution:**

False. $Var(X - Y) = Var(X + (-Y)) = Var(X) + (-1)^2 Var(Y) = Var(X) + Var(Y)$.

8. Fun with Poissons

Let $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$, and X and Y are independent.

(a) Show that $X + Y \sim Poisson(\lambda_1 + \lambda_2)$ **Solution:**

To show that a random variable is distributed according to a particular distribution, we must show that they have the same PMF. Thus, we are trying to show that $P(X + Y = n) = e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!}$

$$
\begin{aligned}
P(X + Y = n) &= \sum_{k=0}^{n} P(X = k \cap Y = n - k) \\
&= \sum_{k=0}^{n} P(X = k) P(Y = n - k) && \text{[X and Y are independent]} \\
&= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k}{k!} \frac{\lambda_2^{n-k}}{(n-k)!} \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n && \text{[Binomial Theorem]}
\end{aligned}
$$

(b) Show that $P(X = k \mid X + Y = n) = P(W = k)$ where $W \sim Bin(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$ **Solution:**

$$
\begin{aligned}
P(X = k \mid X+Y = n) &= \frac{P(X = k \cap X+Y = n)}{P(X+Y = n)} \\
&= \frac{P(X = k \cap Y = n-k)}{P(X+Y = n)} \\
&= \frac{P(X = k)P(Y = n-k)}{P(X+Y = n)} && \text{[X and Y are independent]} \\
&= \frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}} \\
&= \frac{\frac{\lambda_1^k}{k!} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}}{\frac{(\lambda_1+\lambda_2)^n}{n!}} \\
&= \frac{n!}{k!(n-k)!} \cdot \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \\
&= \binom{n}{k} \frac{\lambda_1^k \; \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^k \; (\lambda_1+\lambda_2)^{n-k}} \\
&= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k} \\
&= \binom{n}{k} p^k \; (1-p)^{n-k} \text{ , where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}
$$

9. Memorylessness

We say that a random variable X is memoryless if $\mathbb{P}(X > k+i \mid X > k) = \mathbb{P}(X > i)$ for all non-negative integers k and i. The idea is that X does not *remember* its history. Let $X \sim Geo(p)$. Show that X is memoryless.

Solution:

Let's note that if $X \sim Geo(p)$, then $\mathbb{P}(X > k) = \mathbb{P}(\text{no successes in the first } k \text{ trials}) = (1-p)^k$.

$$
\begin{aligned}
\mathbb{P}(X > k+i \mid X > k) &= \frac{\mathbb{P}(X > k \mid X > k+i)\, \mathbb{P}(X > k+i)}{\mathbb{P}(X > k)} && \text{[Bayes Theorem]} \\
&= \frac{\mathbb{P}(X > k+i)}{\mathbb{P}(X > k)} && [\mathbb{P}(X > k \mid X > k+i) = 1] \\
&= \frac{(1-p)^{k+i}}{(1-p)^k} && [\mathbb{P}(X > k) = (1-p)^k] \\
&= (1-p)^i \\
&= \mathbb{P}(X > i)
\end{aligned}
$$

10. Poisson Practice

Seattle averages 3 days with snowfall per year. Suppose the number of days with snowfall follows a Poisson distribution.

(a) What is the probability of getting exactly 5 days of snow in a year? **Solution:**

Let $X \sim \text{Poi}(3)$ Then $p_X(5) = \frac{3^5 e^{-3}}{5!} \approx .1008$

(b) According to the Poisson model, what is the probability of getting 367 days of snow? **Solution:**

Let $X \sim \text{Poi}(3)$ Then $p_X(367) = \frac{3^{367} e^{-3}}{367!} \approx 1.8 \times 10^{-610}$, that's a very small estimate, but of course the true probability is 0. Recall that using a Poisson distribution is a modeling assumption, it may produce nonzero probabilities for events that are practically impossible.