CSE 312 Section 4

Random Variables and Expectation

Administrivia

Announcements & Reminders

- HW2
  - Grades released on gradescope – check your submission to read comments
  - Regrade requests open ~24 hours after grades are released and close after a week
- HW3
  - Written and Coding due yesterday, Wednesday 7/16 @ 11:59pm
    - Late deadline Friday 7/18 @ 11:59pm
- Quiz 3 Tomorrow!
- Midterm next week (Wednesday 7/23), information on website
- No Homework this week!

Review & Questions

Any lingering questions from this last week?

Each week in section, we'll be reviewing the main concepts from this week and putting them into action by going through some practice problems together. But before we get into that review, we'll try to start off each section with some time for you to ask questions. Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Kahoot for content review!

see task 1 from section handout

Problem 1 – Content Review

1 – Content Review

a) False. It’s a set of all possible values the a random variable can take on.

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b) $Var(X) = \sigma^2$

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b) $Var(X) = \sigma^2$
c) PMF
d) CDF
e) 3/4

1 – Content Review

a) False. It's a set of all possible values the a random variable can take on.
b) $Var(X) = \sigma^2$
c) PMF
d) CDF
e) 3/4
f) 27/16

Problem 2 – Identify that Range!

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

b) The number of lottery tickets I buy until I win it.

c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

Work on this problem with the people around you, and then we'll go over it together!

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

b) The number of lottery tickets I buy until I win it.

Min:

Max:

In between?

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

b) The number of lottery tickets I buy until I win it.

X takes on all positive integer values (I may never win the lottery).

$$\Omega_X = \{1, 2, \dots\}$$

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

Min:

Max:

In between?

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

X takes on every integer value between the min number of heads 0, and the max n.

$$\Omega_X = \{1, 2, \dots, n\}$$

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

Min:

Max:

In between?

2 – Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

Since $\mathbb{P}(\text{head}) = 1$, we are guaranteed to get n heads in n flips..

$$\Omega_X = \{n\}$$

Problem 5 – Hungry Washing Machine

5 – Hungry Washing Machine

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

b) Find $\mathbb{E}[X]$ from the definition of expectation.

c) Find $\mathbb{E}[X]$ using LOE.

Work on this problem with the people around you, and then we'll go over it together!

5 – Hungry Washing Machine

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The washing machine eats 4 socks every time. It can either eat a single sock from 4 pairs of socks, leaving us with 6 complete pairs, or a single sock from 2 pairs and a matching pair, leaving us with 7 complete pairs, or 2 pairs of matching socks, leaving us with 8 complete pairs.

$$\Omega_X = \{6,7,8\}$$

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The washing machine eats 4 socks every time. It can either eat a single sock from 4 pairs of socks, leaving us with 6 complete pairs, or a single sock from 2 pairs and a matching pair, leaving us with 7 complete pairs, or 2 pairs of matching socks, leaving us with 8 complete pairs.

$$\Omega_X = \{6,7,8\}$$

We are dealing with a sample space with equally likely outcomes. As such, we can compute use the formula $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$. We know that $|\Omega| = \binom{20}{4}$ because the washing machine picks a set of 4 socks out of 20 possible socks.

5 – Hungry Washing Machine

a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

To define the pmf of X, we consider each value in the range of X.

For $k = 6$, we first pick 4 out of 10 pairs of socks from which we will eat a single sock ($\binom{10}{4}$ ways), and for each of these 4 pairs we have two socks to pick from ($\binom{2}{1}^4$ ways). Using the product rule, we get $|X = 6| = \binom{10}{4}2^4$

For $k = 7$, we first pick 1 out of 10 pairs of socks to eat in its entirety ($\binom{10}{1}$ ways), and then 2 out of the 9 remaining pairs from which we will eat a single sock ($\binom{9}{2}$ ways), and for each of these 2 pairs we have two socks to pick from ($\binom{2}{1}^2$ ways). Using the product rule, we get $|X = 7| = 10\binom{9}{4}2^2$

For $k = 8$, we pick 2 out of 10 pairs of socks to eat ($\binom{10}{2}$ ways). We get $|X = 8| = \binom{10}{2}$

5 – Hungry Washing Machine

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For $k = 8$, we pick 2 out of 10 pairs of socks to eat ($\binom{10}{2}$ ways). We get $|X = 8| = \binom{10}{2}$

$$p_X(k) = \begin{cases} \frac{\binom{10}{4}2^4}{\binom{20}{4}} & k = 6 \\ \frac{10\binom{9}{4}2^2}{\binom{20}{4}} & k = 7 \\ \frac{\binom{10}{2}}{\binom{20}{4}} & k = 8 \\ 0 & otherwise \end{cases}$$

5 – Hungry Washing Machine

b) Find $\mathbb{E}[X]$ from the definition of expectation.

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$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k) = 6 \cdot \frac{\binom{10}{4} 2^4}{\binom{20}{4}} + 7 \cdot \frac{10 \binom{9}{2} 2^2}{\binom{20}{4}} + 8 \cdot \frac{\binom{10}{2}}{\binom{20}{4}} = \frac{120}{19}$$

5 – Hungry Washing Machine

c) Find $\mathbb{E}[X]$ using LOE.

5 – Hungry Washing Machine

c) Find $\mathbb{E}[X]$ using linearity of expectation.

For $i \in [10]$, let X_i be 1 if pair i survived, and 0 otherwise. Then, $X = \sum_{i=1}^{10} X_i$.
Now, we can calculate $\mathbb{E}[X_i]$

$$\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = \frac{\binom{18}{4}}{\binom{20}{4}}$$

$\binom{18}{4}$ Indicates the number of ways of choosing 4 out of the 18 remaining socks (we spare our chose pair i)

Then, by LOE

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} \mathbb{E}[X_i] = 10 \cdot \mathbb{E}[X_i] = \frac{120}{19}$$

Midterm Review!

Problem 4 – Kit Kats Again

4 – Kit Kats Again

We have N candies in the jar.
We have K kit kats in the jar.

We are drawing without replacement until we have k kit kats.

$$k \leq K \leq N$$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?
What is $p_X(n) = \mathbb{P}(X = n)$?

Work on finding the range of X with the people around you!

4 – Kit Kats Again

We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?

Min:
Max:
In between:

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What is Ω_X, the range of X?

Min: **k** (we were just really lucky and kept picking kit kats one after the other until we had all k that we wanted)
Max:
In between:

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We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?

Min: **k** (we were just really lucky and kept picking kit kats one after the other until we had all k that we wanted)
Max: **N – K + k** (ended up pick out all the non kit kats (N - K) in the process of picking out our k kit kats)
In between:

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We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X?

Min: **k** (we were just really lucky and kept picking kit kats one after the other until we had all k that we wanted)

Max: **N – K + k** (ended up pick out all the non kit kats (N - K) in the process of picking out our k kit kats)

In between: Any integer value in between

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What is Ω_X, the range of X?

$$\Omega_X = \{k, k+1, \ldots, N-K+k\}$$

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What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Work on finding the PMF of X with the people around you!

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What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \ldots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Consider all N candies to be arranged randomly in a row. This is the order in which we will be drawing our n candies.

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Consider all N candies to be arranged randomly in a row. This is the order in which we will be drawing our n candies.

We know that the nth candy is a kitkat (the kth kitkat) to be chosen. This nth candy divides the row into 2 sections.

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What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Left: (?) candies and (?) kit kats
nth spot: kth kit kat
Right: (?) candies and (?) kit kats

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What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \ldots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Left: **n - 1** candies and **k - 1** kit kats
nth spot: kth kit kat
Right: (?) candies and (?) kit kats

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What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$

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Left: **n - 1** candies and **k - 1** kit kats

nth spot: kth kit kat

Right: **N - n** candies and **K - k** kit kats

Event:

Sample Space:

4 – Kit Kats Again

We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Left: **n - 1** candies and **k - 1** kit kats
nth spot: kth kit kat
Right: **N - n** candies and **K - k** kit kats

Event: number of ways to arrange the candies to have this outcome
Sample Space: number of ways to arrange the candies

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Let X be the number of draws until the kth kit kat (this includes the kth kit kat).

What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$
What is $p_X(n) = \mathbb{P}(X = n)$?

Left: **n - 1** candies and **k - 1** kit kats
nth spot: kth kit kat
Right: **N - n** candies and **K - k** kit kats

|E|: $\binom{n-1}{k-1}\binom{N-n}{K-k}$
|S|: $\binom{N}{K}$

4 – Kit Kats Again

We have N candies in the jar. We have K kit kats in the jar. We are drawing without replacement until we have k kit kats. $k \leq K \leq N$

Let X be the number of draws until the kth kit kat (this includes the last kit kat).

What is Ω_X, the range of X? $\Omega_X = \{k, k+1, \dots, N-K+k\}$

What is $p_X(n) = \mathbb{P}(X = n)$? $p_X(n) = \mathbb{P}(X = n) = \frac{\binom{n-1}{k-1}\binom{N-n}{K-k}}{\binom{N}{K}}$ if $n \in \Omega_X$ and 0 otherwise

Justification: Consider all N candies to be arranged randomly in a row. We will treat the first n candies to be the "chosen" candies. We know that the nth candy is a kitkat (the kth kitkat) to be chosen. This nth candy divides the row into 2 sections. On the left, are the $n-1$ candies that were chosen (all candies chosen except the last kitkat). On the right, are the candies remaining in the jar. So the first term in the numerator is choosing the $k-1$ spots where kitkats will be chosen from the $n-1$ spots. The second term is choosing the $K-k$ spots for the kitkats that remain in the jar among the remaining $N-n$ candies. The denominator is the total number of possible ways to place the K kitkats among N candies.

Problem 7 – Frogger

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A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let X be the location of the frog.

a) Find $p_X(k)$, the probability mass function for X.

b) Compute $\mathbb{E}[X]$ from the definition.

Work on this problem with the people around you, and then we'll go over it together!

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a) Find $p_X(k)$, the probability mass function for X.

Let L be a left step, R be a right step, and N be no step.

The range of X is $\{-2, -1, 0, 1, 2\}$. We can compute:

$p_X(-2) = \mathbb{P}(X = -2) = \mathbb{P}(LL) = p_2^2$

$p_X(-1) = \mathbb{P}(X = -1) = \mathbb{P}(LN \cup NL) = 2p_2p_3$

$p_X(0) = \mathbb{P}(X = 0) = \mathbb{P}(NN \cup LR \cup RL) = p_3^2 + 2p_1p_2$

$p_X(1) = \mathbb{P}(X = 1) = \mathbb{P}(RN \cup NR) = 2p_1p_3$

$p_X(2) = \mathbb{P}(X = 2) = \mathbb{P}(RR) = p_1^2$

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$p_X(2) = \mathbb{P}(X = 2) = \mathbb{P}(RR) = p_1^2$

$$p_X = \begin{cases} p_2^2 & k = -2 \\ 2p_2p_3 & k = -1 \\ p_3^2 + 2p_1p_2 & k = 0 \\ 2p_1p_3 & k = 1 \\ p_1^2 & k = 2 \\ 0 & otherwise \end{cases}$$

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b) Compute $\mathbb{E}[X]$ from the definition.

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b) Compute $\mathbb{E}[X]$ from the definition.

$$\mathbb{E}[X] = (-2)(p_2^2) + (-1)(2p_2p_3) + (0)(p_3^2 + 2p_1p_2) + (1)(2p_1p_3) + (2)(p_1^2) = 2(p_1 - p_2)$$

That’s All, Folks!

Thanks for coming to section this week!
Any questions?