CSE 312 Section 2

Intro to Discrete Probability

Administrivia

Announcements & Reminders

- Office Hours
  - Offered every day!
  - Times posted on the calendar on the course website

- HW1
  - Was due yesterday, Wednesday 7/2
  - Late deadline Saturday 7/4 (with meaningful attempt submitted yesterday)

- HW2
  - Released on the course website
  - Due Wednesday 7/9

Review

What have we learned?

- Principle of Inclusion-Exclusion
- Stars and Bars
- Pigeonhole principle
- Discrete Probability
- Conditional Probability

Any lingering questions from this last week?

Problem 1 – Content Review

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g) False (Correct statement is $P(A \cup B) \leq P(A) + P(B)$)

Problem 2 – A Team and a captain

6 – Powers and Divisibility

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r} r$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members)

Work on this problem with the people around you, and then we'll go over it together!

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LHS: First, choose a captain (n choices). Then, we choose the remaining r-1 team members from the pool of remaining n-1 people.

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LHS: First, choose a captain (n choices). Then, we choose the remaining r-1 team members from the pool of remaining n-1 people ($\binom{n-1}{r-1}$ choices).

RHS: First, choose the team of r members ($\binom{n}{r}$ choices). Then, choose the captain of this given team (r choices). Apply product rule.

Problem 6 – Powers and Divisibility

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HINT: think about what are the pigeons, and what are the pigeonholes?

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Each of which can be expressed as $7^i = 2003x_i + r_i$.

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Therefore, by Pigeonhole principle, there exists a pair of members, say 7^n and 7^m, where $n > m$, such that $r_n = r_m$.

Then their difference $7^n - 7^m = 2003x_n + r_n - 2003x_m - r_m = 2003(x_n - x_m)$ is divisible by 2003.

Problem 10 – Balls from an Urn

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Say an urn contains one red ball, one blue ball, and one green ball. (Other than for their colors, balls are identical.) Imagine we draw two balls with replacement, i.e., after drawing one ball, with put it back into the urn, before we draw the second one. (In particular, each ball is equally likely to be drawn.)

a) Give a probability space describing the experiment.
b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)
c) What is the probability that at most one ball is red?
d) What is the probability that we get at least one green ball?
e) Repeat b)-d) for the case where the balls are drawn without replacement, i.e., when the first ball is drawn, it is not placed back from the urn.

Work on this problem with the people around you, and then we'll go over it together!

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$\Omega = \{BB, BR, BG, RB, RR, RG, GB, GR, GG\} = \{B, R, G\}^2$

$\mathbb{P}(\omega) = 1/9 \text{ for all } \omega \in \Omega$

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The event is $B = \{RR\}$

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The event is $B = \{RR\}$

The probability of the event is $\mathbb{P}(B) = \frac{|B|}{9} = 1/9$

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The event is $C = \{BB, BR, BG, RB, RG, GB, GR, GG\}$

The probability of the event is $\mathbb{P}(C) = \frac{|C|}{9} = 8/9$

Alternatively, this is just B^C, the complement of B.

We know that $\mathbb{P}(B^C) = 1 - \mathbb{P}(B) = 1 - 1/9 = 8/9$

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The event is $D = \{BG, GB, RG, GR, GG\}$

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e) Repeat b)-d) for the case where **the balls are drawn without replacement.**

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      $\mathbb{P}(\omega) = \frac{1}{3} \cdot \frac{1}{2} = 1/6$ for all $\omega \in \Omega$ (3 choices for the first ball, 2 for the second)

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$D = \{BG, GB, RG, GR\}$ so $\mathbb{P}(D) = \frac{4}{6} = \frac{2}{3}$

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- $\mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6)$
- $\mathbb{P}(1) = 3 \cdot \mathbb{P}(3)$

What is the probability that the outcome is 3 or 4?

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Since $\mathbb{P}(1) = \mathbb{P}(2)$ and $\mathbb{P}(1) = 3 \cdot \mathbb{P}(3)$, we have that:

$$1 = \mathbb{P}(1) + \mathbb{P}(2) + \mathbb{P}(3) + \mathbb{P}(4) + \mathbb{P}(5) + \mathbb{P}(6)$$
$$= 3 \cdot \mathbb{P}(3) + 3 \cdot \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) = 10 \cdot \mathbb{P}(3)$$

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Thus, solving algebraically, $\mathbb{P}(3) = 0.1$, so $\mathbb{P}(3) = \mathbb{P}(4) = 0.1$.

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Thus, solving algebraically, $\mathbb{P}(3) = 0.1$, so $\mathbb{P}(3) = \mathbb{P}(4) = 0.1$.
Since rolling a 3 and 4 are disjoint events, then $\mathbb{P}(3 \text{ or } 4) = \mathbb{P}(3) + \mathbb{P}(4) = 0.1 + 0.1 = 0.2$

Problem 2 - Subsubset

2 – Subsubset

- Let $[n] = \{1,2,\ldots,n\}$ denote the first n natural numbers. How many (ordered) pairs of subsets (A,B) are there such that $A \subseteq B \subseteq [n]$?

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Realize that, if there are no restrictions, for each element i of $1,\dots,n$, there are four possibilities: it can be in only A, only B, both, or neither. In our case, there is only one that is not valid (violates $A \subseteq B$): being in A but not B. Hence there are 3 choices for each element, so the total number of such ordered pairs of subsets is

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$$3^n$$

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Alternately, apply the sum rule by adding up the number of ways of doing this where B has size k, where k is any integer between 0 and n. Now apply the product rule to find the number of ways to choose B of size exactly k (there are $\binom{n}{k}$ possibilities for B), and then once B is selected, count the number of ways of choosing A which has to be a subset of B (2^k ways).

Hence, by the Binomial Theorem, the number of such ordered pairs of subsets is

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$$\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = 3^n$$

Problem 4 – GREED INNIT

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Find the number of ways to rearrange the word “INGREDIENT”, such that no two identical letters are adjacent to each other. For example, “INGREEDINT” is invalid because the two E’s are adjacent.

Repeat the question for the letters “AAAAABBB”.

Work on this problem with the people around you, and then we’ll go over it together!

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By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ = singles – doubles + triples, and by complementing, $|\Omega \backslash (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

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Singles = $|A_I| + |A_E| + |A_N|$

Doubles = $|A_I \cap A_E| + |A_I \cap A_N| + |A_E \cap A_N|$

Triples = $|A_I \cap A_E \cap A_N|$

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4 – GREED INNIT

Find the number of ways to rearrange the word “INGREDIENT”, such that no two identical letters are adjacent to each other.

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$|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I’s as one entity.
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Putting this together gives $\frac{10!}{2!2!2!} - \left(3 \cdot \frac{9!}{2!2!} - 3 \cdot \frac{8!}{2!} + 7!\right)$

4 – GREED INNIT

Repeat the question for the letters “AAAAABBB”.

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Note that no A's and no B's can be adjacent. So let us put the B's down first:

_B_B_B_

By the pigeonhole principle, two A's must go in the same slot, but then they would be adjacent, so there are no ways .

Problem 12 – Shuffling Cards

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We have a deck of cards, with 4 suits, with 13 cards in each. Within each suit, the cards are ordered Ace > King > Queen > Jack > 10 > · · · > 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely). What is the probability the first card on the deck is (strictly) larger than the second one?

Work on this problem with the people around you, and then we'll go over it together!

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First off, the sample space Ω here consists of all pairs of cards – which we can represent by their value and suit, e.g., $(4\clubsuit, A\diamondsuit)$. There are $52 \cdot 51 = 2652$ possible outcomes, therefore $\mathbb{P}(\omega) = \frac{1}{2652}$ for all $\omega \in \Omega$.

Let us now look at the size of the event E containing all pairs where the first card is strictly larger than the second. Then, the number of pairs of values of cards a and b where $a < b$ is exactly $\binom{13}{2} = 13 \cdot 6 = 78$. We can then assign suits to each of them – given the cards are different, all suits are possible for each, so there are $4^2 = 16$ choices. Thus, overall,

$$|E| = 16 \cdot 78 = 1248$$

Therefore, $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{16 \cdot 78}{52 \cdot 51} = \frac{13 \cdot 3 \cdot 2^5}{13 \cdot 3 \cdot 2^2 \cdot 17} = \frac{8}{17} \approx 0.47$

Problem 5 - Friendships

5 – Friendships

- Show that in any group n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.) Solve in particular the following two cases individually:

a) Everyone has at least one friend.

b) At least one person has no friends.

Work on this problem with the people around you, and then we'll go over it together!

5 – Friendships

a) Everyone has at least one friend.

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Everyone has between 1 and $n-1$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, two of them will have the same number of friends.

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b) At least one person has no friends.

Here, we need to observe that if someone has 0 friends, then nobody has $n-1$ friends (by the symmetry of the friendship relation). Then, possible choices are now between 0 and $n-2$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, two of them will have the same number of friends.

That’s All, Folks!

Thanks for coming to section this week!
Any questions?