CSE 312 Section 1

Combinatorics

Administrivia & Introductions

Announcements & Reminders

- Office Hours
  - We started holding office hours today!
  - Times posted on the calendar on the course website

- HW1
  - Due Wednesday 7/2 @ 11:59pm

- Quiz 1
  - Tomorrow in lecture, start at 12:00, 15 mins long
  - Covers Monday and Wednesday content

Section

- Materials
    - Handouts will be provided in at each section
    - Worksheets and sample solutions will be available on the course calendar later after section
- Participation
    - Participations are recorded
    - "Participated" means you're working on the problems and talking with those around you.
    - If you cannot attend section in-person, do the section problems and email your TA(s) for credit
    - **Section {AA/AB}: email: …**

Homework

- Submissions
  - LaTeX (highly encouraged)
    - overleaf.com
    - template and LaTeX guide posted on course website!
  - Word Editor that supports mathematical equations
  - Handwritten neatly and scanned
- Homework will typically be due on Wednesdays at 11:59pm on Gradescope

Homework

- Late Days
  - 2 penalty free late days with meaningful attempt submitted by the original deadline
  - Meaningful Attempt
    - You have thought about and worked through the majority of every problems on the homework
  - Failing to submit a meaningful attempt would result in 25% deduction per day after original deadline

Your TAs

- TA 1
  - Intro
- TA 2
  - Intro

Our Office Hours

- TA 1
  - OH
- TA 2
  - OH

Icebreaker

- Small groups of 4-6ish
- Please share with your group
  - Your name
  - Number of years in department/ at UW
  - What was something fun you did over Winter break?
  - What are you concerned about for 312 / what are you excited about?
- Then, share how you like to eat your potatoes (baked, fried, chips,
- We'll go around and see what style of potato is most popular!

Review

What have we learned so far?

- Sum Rule
- Product Rule
- Picking objects
  - Permutations: $P(n,k) = \frac{n!}{(n-k)!}$
  - Combinations: $C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Overcounting
- Complementary counting

Any lingering questions from this last week?

Each week in section, we'll be reviewing the main concepts from this week by going through some practice problems together. But before that, we'll try to start off each section with some time for you to ask questions.

Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Sum and Product Rule

- Sum Rule
  - An experiment can **either** end up being one of N outcomes, **or** one of M outcomes (where there is no overlap)
  - The total number of possible outcomes is **N + M.**
  - Outcomes: A, B, C
  - Total outcomes: $A \cup B \cup C$

- Product Rule
  - Sequential process, where step i has n_i options
  - For k steps, The total number of possible outcomes is $\prod \boldsymbol{n_i} = \boldsymbol{n_1} \cdot \boldsymbol{n_2} \cdots \boldsymbol{n_k}$

Picking objects

- Picking k objects from n objects

- Permutations
  - Arranging k objects: Order matters
  - $P(n,k) = \frac{n!}{(n-k)!}$

- Combinations
  - Choosing k objects: Order doesn’t matter
  - $C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Picking objects

- Picking k objects from n objects

		Repeats	
		with	without
Order	matters	n^k	$P(n,k)$
	Doesn't matter		$C(n,k)$

Overcounting

- Example from class

Overcounting

How many anagrams are there of RHUBARB 

Final answer $\frac{7!}{2!\cdot 2!}$

One more piece of notation – “multinomial coefficient”

$\binom{7}{2,2}$ is alternate notation for $\frac{7!}{2!2!}$.

In general: $\binom{n}{k_1,k_2,\ldots,k_\ell} = \frac{n!}{k_1!\cdot k_2!\cdots k_\ell!}$

Popular notation among mathematicians.

Complementary Counting

- All possible outcome is Ω
- We want $A \subseteq \Omega$
- It might be hard to count the number of outcome in A. Instead, it could be easier to count $\overline{A}$.

$$|A| = |\Omega \setminus \overline{A}|$$

Problem 9 – Rabbits!

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Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

Work on this problem with the people around you, and then we'll go over it together!

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Use the sum rule:

1. Peter and Pauline go to the same store
2. Peter and Pauline go to different stores

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1. Peter and Pauline go to the same store

Step 1: There are 4 stores it could be.
Step 2: For each choice, there are 3 choices of stores for each of the 3 offspring.
So, the number of ways to distribute rabbits in this case is:

2. Peter and Pauline to to different stores

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1. Peter and Pauline go to the same store: $4 \cdot 3^3$
2. Peter and Pauline to to different stores

Step 1: There are $4 \cdot 3 = 12$ pairs of stores they could go to.
Step 2: Then for the 3 offspring, there are 2 choices of store for each of them
Therefore, the number of ways in this case is:

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Step 1: There are $4 \cdot 3 = 12$ pairs of stores they could go to.
Step 2: Then for the 3 offspring, there are 2 choices of store for each of them.
Therefore, the number of ways in this case is:

$$12 \cdot 2^3$$

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1. Peter and Pauline go to the same store: $4 \cdot 3^3$
2. Peter and Pauline to to different stores: $12 \cdot 2^3$

Using sum rule, the answer is

$$4 \cdot 3^3 + 12 \cdot 2^3$$

Problem 5 – Birthday Cake

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A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

Work on this problem with the people around you, and then we'll go over it together!

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Then, given the choice on Wednesday, there are 4 choices for Tuesday, and given the choice on Tuesday, there are 4 choices for Monday, and given the choice on Monday, there are 4 choices on Sunday. Similarly, given the choice on Friday, there are 4 choices on Saturday.

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Therefore, the answer is: $4 \cdot 4 \cdot 4 \cdot 4 \cdot 1 \cdot 4 \cdot 4 = 4^6$

Problem 1 - Seating

1 – Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if …

a) … all couples are to get adjacent seats?
b) … anyone can sit anywhere, except that one couple insists on not sitting in adjacent seats?

Work on this problem with the people around you, and then we'll go over it together!

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Apply the product rule, first choosing one of the 5! permutations of the 5 couples, and then, for each couple in turn, choosing one of the 2 permutations for how they sit (for a total of 2^5). Therefore, the answer is:

$$5! \cdot 2^5$$

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One way we can solve this is to consider by cases.

Name the two people in the couple A and B. There are two cases: A can sit on one of the ends, or not.

1 – Seating

b) How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if anyone can sit anywhere, except that one couple insists on not sitting in adjacent seats?

One way we can solve this is to consider by cases.

Name the two people in the couple A and B. There are two cases: A can sit on one of the ends, or not.
Case 1: If A sits on an end seat, A has 2 choices and B has 8 possible seats.
Case 2: If A doesn't sit on the end, A has 8 choices and B only has 7.
So, there are a total of $2 \cdot 8 + 8 \cdot 7$ ways A and B can sit.

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Once they do, the other 8 people can sit in 8! ways since there are no other restrictions.

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Once they do, the other 8 people can sit in 8! ways since there are no other restrictions.
Hence the total number of ways is

$$(2 \cdot 8 + 8 \cdot 7)8! = 9 \cdot 8 \cdot 8! = 8 \cdot 9!$$

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b) How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if anyone can sit anywhere, except that one couple insists on not sitting in adjacent seats?

Another way to solve this is to apply complementary counting to first compute the total number of arrangements of the 10 people, and then subtract from this the number of arrangements in which that particular couple does get adjacent seats.

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Total arrangements: 10!

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Ways for a particular couple to get adjacent seats. Since you can treat the couple as a unit, permute the 9 “individuals” (consisting of 8 people plus the couple) and then consider the 2 permutations for that couple: $9! \cdot 2$

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Total arrangements: 10!
Ways for a particular couple to get adjacent seats. Since you can treat the couple as a unit, permute the 9 "individuals" (consisting of 8 people plus the couple) and then consider the 2 permutations for that couple: $9! \cdot 2$

That means the answer to the question is

$$10! - 9! \cdot 2 = 9!\,(10 - 2) = 8 \cdot 9!$$

Problem 4 – Escape the Professor

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There are 6 security professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are chosen and paired off, how many pairings are possible?

Work on this problem with the people around you, and then we'll go over it together!

4 – Escape the Professor

There are 6 security professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are chosen and paired off, how many pairings are possible?

Apply the product rule to first choose 4 of the security professors, then 4 of the theory professors. Then assign each theory professor to a security professor (4 choices for the first, 3 for the second and so on).

The answer is

4 – Escape the Professor

There are 6 security professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are chosen and paired off, how many pairings are possible?

Apply the product rule to first choose 4 of the security professors, then 4 of the theory professors. Then assign each theory professor to a security professor (4 choices for the first, 3 for the second and so on).

The answer is

$$\binom{6}{4} \cdot \binom{7}{4} \cdot 4!$$

Problem 11 - Subsubset

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Let $[n] = \{1, 2, ..., n\}$ denote the first n natural numbers. How many (ordered) pairs of subsets (A, B) are there such that $A \subseteq B \subseteq [n]$?

Work on this problem with the people around you, and then we'll go over it together!

11 – Subsubset

Let $[n] = \{1, 2, ..., n\}$ denote the first n natural numbers. How many (ordered) pairs of subsets (A, B) are there such that $A \subseteq B \subseteq [n]$?

Realize that, if there are no restrictions, for each element i of $1, \ldots, n$, there are four possibilities: it can be in only A, only B, both, or neither. In our case, there is only one that is not valid (violates $A \subseteq B$): being in A but not B. Hence there are 3 choices for each element, so the total number of such ordered pairs of subsets is

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$$3^n$$

Problem 7 – Paired Finals

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Suppose you are to take a CSE 312 final in pairs. There are 100 students in the class and 8 TAs, so 8 lucky students will get to pair up with a TA. Each TA must take the exam with some student, but two Tas cannot take the exam together. How many ways can they pair up?

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Apply the product rule.
Step 1: Choose 8 lucky students, pair them with a TA. There are $\binom{100}{8}$ ways.

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Step 2: Pairing these 8 students with 8 TAs. There are $8!$ ways.

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Apply the product rule.
Step 1: Choose 8 lucky students, pair them with a TA. There are $\binom{100}{8}$ ways.
Step 2: Pairing these 8 students with 8 TAs. There are $8!$ ways.
Step 3: Paring other 92 students

The first one has 91 choices. Then there are 90 students left.
So, the next one has 89 choices.
And so on.
Therefore, there are $91 \cdot 89 \cdots 3 \cdot 1$ ways

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Step 1: Choose 8 lucky students, pair them with a TA. There are $\binom{100}{8}$ ways.
Step 2: Pairing these 8 students with 8 TAs. There are $8!$ ways.
Step 3: Paring other 92 students. There are $91 \cdot 89 \cdots 3 \cdot 1$ ways

Using product rule, the answer is:

$$\binom{100}{8} \cdot 8! \cdot 91 \cdot 89 \cdots 3 \cdot 1$$

Problem 10 – Extended Family Portrait

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A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together?

Work on this problem with the people around you, and then we'll go over it together!

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First order the families; there are $n!$ ways to do this.

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First order the families; there are $n!$ ways to do this.

Then consider each of the n families one by one and reorder their members. Within each family, there are $m!$ ways to order their members.

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First order the families; there are $n!$ ways to do this.

Then consider each of the n families one by one and reorder their members. Within each family, there are $m!$ ways to order their members.

So, the total number of ways to line these people up according to the given constraints is

$$n!\,(m!)^n$$

That’s All, Folks!

Thanks for coming to section this week!
Any questions?