Section Midterm: Solutions

1. True or False?

Let A and B be events in the same sample space that each have non-zero probability. For each of the following statements, state whether it is always true, always false, or it depends on information not given.

(a) If A and B are mutually exclusive, then they are independent.

Solution:

False, $\mathbb{P}(A \cap B) = 0 \neq \mathbb{P}(A)\mathbb{P}(B)$.

(b) If A and B are independent, then they are mutually exclusive.

Solution:

False, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \neq 0$.

(c) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are mutually exclusive.

Solution:

False, because then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) = 1.5 > 1$, which is impossible.

(d) If $\mathbb{P}(A) = \mathbb{P}(B) = 0.75$, then A and B are independent.

Solution:

Depends whether $\mathbb{P}(A \cap B) = \frac{9}{16}$.

2. Question 2

Given any set of 18 integers, show that one may always choose two integers so that their difference is divisible by 17.

Solution:

Note that there are 17 possible remainders when dividing an integer by 17. Let 18 integers be pigeons, and let 17 possible remainders be pigeonholes. By the pigeonhole principle, we know that there exist at least 2 integers that must have the same remainder when divided by 17. Call them x and y. Then, formally $x \equiv y \pmod{17}$. By equivalence in modular arithmetic, $17 | (x - y)$.

3. Count the Solutions 2

Consider the following inequality: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \leq 70$. A solution to this inequality over the non-negative integers is a choice of a non-negative integer for each of the 6 variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the inequality. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the inequality?

Solution:

We introduce a new variable a_7 here. Then, the question is equivalent to asking how many different solutions there are to the following equality

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 70$$

Think of splitting 70 1's into 7 blocks. The number of 1's in the i-th block corresponds to the value of a_i. We see that this is a stars and bars problem where we have $n = 70, k = 7$. Then the answer is $\binom{76}{6}$

4. Question 4

You roll 3 fair dice, each with a different numbers of faces: die 1 has six faces (numbered $1 \ldots 6$), die 2 has eight faces (numbered $1 \ldots 8$), and die 3 has twelve faces (numbered $1 \ldots 12$). Let the random variable X be the sum of the three values rolled. What is $\mathbb{E}[X]$?

Solution:

Let D_1, D_2, D_3 be the values of die 1, die 2, and die 3, respectively. $\mathbb{E}[D_1] = 3.5$, $\mathbb{E}[D_1] = 4.5$, and $\mathbb{E}[D_1] = 6.5$. Then $X = D_1 + D_2 + D_3$, Therefore, by linearity of expectation, $\mathbb{E}[X] = \mathbb{E}[D_1 + D_2 + D_3] = \mathbb{E}[D_1] + \mathbb{E}[D_2] + \mathbb{E}[D_3] = 14.5$.

5. Question 5

How many integers in $\{1, 2, \ldots, 360\}$ are divisible by one or more of the numbers 2, 3 and 5?

Solution:

Let A be a set of integers divisible by 2, B be a set of integers divisible by 3, C be a set of integers divisible by 5.

We wish to find $|A \cup B \cup C|$. By principle of inclusion-exclusion, we have:

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= \frac{360}{2} + \frac{360}{3} + \frac{360}{5} - \frac{360}{6} - \frac{360}{10} - \frac{360}{15} + \frac{360}{30} \\ &= 264\end{aligned}$$

Note on calculating the size of intersection of sets: for example, $A \cap B$ is a set of integers divisible by 2 and 3, i.e. divisible by 6. Then, it's the same for $A \cap C, B \cap C, A \cap B \cap C$.

6. Question 6

Suppose a special deck has 4 suits with 5 cards in each suit. It is shuffled well and then dealt into 5 piles of 4 cards each. Let E_i refer to the event that pile i has exactly one spade. Compute the probability $\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5)$

Solution:

$$\frac{20 \cdot 16 \cdot 12 \cdot 8 \cdot 4}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}$$

Think of laying out all 20 cards out in order. Then, the first 4 cards make up for 1st pile, next 4 cards make up for 2nd pile, and so on. In this way, we see that there are 20 positions in which we can put our cards. For numerator, we calculate the probability by calculating the number of ways we can put our spades such that each pile has 1 spade. First, there are 20 places to put our first spade. Then, there are 16 places to place the next spade because we can't put it anywhere within 4 cards that make up the first pile. Then, there are 12, 8, and 4 places to put the 3rd, 4th, and 5th spade respectively. For denominator, there are $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$ ways to put 5 spades.

7. Question 7

You are trying to diagnose the probability that a patient with a positive blood sugar test result has diabetes, even though she is in a low risk group. The probability for women in this group having diabetes is 0.8%. 90% of women with diabetes will test positive in the blood sugar test. 7% of women without diabetes will test positive in the blood sugar test. Your patient tests positive in the blood sugar test. What is the probability that she has diabetes?

Solution:

Let D be the event that she has diabetes and T be the event that she is tested positive.

$$\begin{aligned}\mathbb{P}(D|T) &= \frac{\mathbb{P}(T|D)\mathbb{P}(D)}{\mathbb{P}(T)} \\ &= \frac{\mathbb{P}(T|D)\mathbb{P}(D)}{\mathbb{P}(T|D)\mathbb{P}(D) + \mathbb{P}(T|D^C)\mathbb{P}(D^C)} \\ &= \frac{0.9 \cdot 0.008}{0.9 \cdot 0.008 + 0.07 \cdot 0.992} \\ &\approx 0.009\end{aligned}$$

Notice that the posterior probability 0.09 of diabetes is approximately 10 times as great as the prior probability 0.008 of diabetes, but still small.

8. Question 8

A very long multiple choice exam has 4 choices for each questions. Charlie has studied enough so that he know the correct anser for $\frac{1}{2}$ of the questions; for an additional $\frac{1}{4}$ of the questions he can eliminate one choice and chooses randomly and uniformly among the other three, and for the remaining $\frac{1}{4}$ of the questions he chooses randomly and uniformly among all four answers.

As the teacher, you want to determine how many answers the student actually knows. For a randomly chosen question, if Charlie answers it correctly, what is the probability he knew the answer?

Solution:

Let C be the event that Charlie has the correct answer and K be the event that Charlie knew the answer.

Then by LTP, we can calculate:

$$P(C) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{4} = \frac{31}{48}$$

Now we can calculate the probability that he knew the answer given that he answered it correctly using Bayes' theorem as follows

$$\begin{aligned}\mathbb{P}(K|C) &= \frac{\mathbb{P}(C|K)\mathbb{P}(K)}{\mathbb{P}(C)} \\ &= \frac{1 \cdot \frac{1}{2}}{\frac{31}{48}} \\ &= \frac{24}{31}\end{aligned}$$

9. Question 9

The space shuttle has 6 O-rings: these were invloved in the Challenger disaster. When the space hsuttle is launched, each O-ring has a probability of failure of 0.0137, independent of whether other O-rings fail.

(a) What is the probability that, during 23 launches, no O-ring will fail, but that at least one O-ring will fail during the 24th launch?

Solution:

The probability that no O-ring fails on a single launch is $(1-0.0137)^6 \approx 0.921$. The probability that this happen for 23 launches and doesn't happen on the 24th launch is $0.921^{23}(1-0.921) \approx 0.0118$.

(b) What is the probability that no O-ring fails during 24 launches?

Solution:

$0.921^{24} \approx 0.137$

10. Question 10

Suppose you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur? Assume that there are 365 possible birthdays and each one is equally probable for a randomly chosen person.

Solution:

To take exactly 20 people for this to occur, the first 19 people must not have the same birthday as your own. The probability of this is $\left(\frac{364}{365}\right)^{19}$. Then, for the 20th person to have the same birthday as you, the probability of this is $\frac{1}{365}$. Combined, the answer is $\left(\frac{364}{365}\right)^{19} \cdot \frac{1}{365}$.

11. Question 11

Two fair 6-sided dice are thrown n times in succession.

(a) Compute the probability that double 6 (i.e., 6 on each die) appears at least once in the n throws.

Solution:

The probability that double 6 does not occur in n throws is $\left(\frac{35}{36}\right)^n$. Therefore, the complement of it, double 6 appears at least once is $1 - \left(\frac{35}{36}\right)^n$.

(b) How large does n need to be to make this probability at least $\frac{1}{2}$?

Solution:

Solving $1 - \left(\frac{35}{36}\right)^n \geq \frac{1}{2}$ gives us $n \geq 24.6$. Since n must be integer, $n \geq 25$.

12. Question 12

You are working on a difficult passage from a new piece you are learning on the piano. You wish to play it correctly 4 times before stopping for the day. If your probability of playing it correctly on each attempt is $\frac{2}{3}$, and the attempts are independent (unfortunately!), what is the probability that you have to play it at least 8 times?

Solution:

This is equivalent to asking the probability that, in the first 7 attempts, you play it correctly 3 or fewer times. There are 4 cases here:

(a) You don't get any attempts correct. The probability of this is $\left(\frac{1}{3}\right)^7$.

(b) You play it correctly 1 time. The probability of this is $\binom{7}{1} \cdot \left(\frac{2}{3}\right)^1 \cdot \left(\frac{1}{3}\right)^6$.

(c) You play it correctly 2 times. The probability of this is $\binom{7}{2} \cdot \left(\frac{2}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^5$.

(d) You play it correctly 3 times. The probability of this is $\binom{7}{3} \cdot \left(\frac{2}{3}\right)^3 \cdot \left(\frac{1}{3}\right)^4$.

Summing all these up, we get:

$$\frac{379}{3^7} \approx 0.173$$

13. Cash???

The probability that a customer pays with cash is 40%, independent of other customers. Find the probability that the 12th customer to arrive at the cashier is the 8th one that pays with cash.

Solution:

This is equivalent to asking the probability that, in the first 11 customers, 7 customers pay with cash and the 12th customer to pay with cash. First, we calculate the probability that 7 out of first 11 customers pay with cash. That is $\binom{11}{7} \cdot (0.4)^7 \cdot (0.6)^4$. Next, the probability that the 12th customer pays with cash is 0.4. Together, we get the answer to be $\binom{11}{7} \cdot (0.4)^8 \cdot (0.6)^4$

14. Question 14

Let X be the outcome of rolling a fair 6-sided die once. Let Y be the sum of the outcomes of rolling the same die n times independently.

(a) Compute $\mathbb{E}[X]$.

Solution:

$\mathbb{E}[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$

(b) Compute $Var(X)$ and the standard deviation σ of X.

Solution:

$$\mathbb{E}[X^2] = 1 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = \frac{91}{6}$$
$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{91}{6} - (3.5)^2 = \frac{35}{12}$$
$$\sigma = \sqrt{Var(X)} = \sqrt{\frac{35}{12}} \approx 1.7$$

(c) Compute $\mathbb{E}[Y]$.

Solution:

For $1 \leq i \leq n$, let X_i be the outcome of the i-th die roll.

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}\left[X_i\right] = \sum_{i=1}^{n} \frac{7}{2} = \frac{7}{2}n$$

(d) Compute *Var*(Y)

Solution:

Because $X_1, X_2, \ldots, X_n$ are independent,

$$Var(Y) = Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \frac{35}{12} = \frac{35}{12}n$$

15. Lucky Bomb Defusal

You are in a tense bomb-defusal scenario with three wires: Red (R), Yellow (Y), and Blue (B). There is a single 'lucky' wire that will defuse the bomb, the other two will cause it to explode.

Captured, the bomber has promised to help, albeit deceitfully. If you pretend to cut a wire, the bomber will reveal a non-lucky (incorrect) wire from the remaining two. If both remaining wires are non-lucky they will pick randomly.

Suppose that you pretend to cut wire R. The bomber then reveals that the lucky wire is not wire B. Given this information, should you cut wire R or wire Y?

Let $\overline{B}$ be the event that the bomber Reveals B to be the incorrect choice.

Let R be the event that the 'lucky'/correct wire is R.

Let Y be the even that the 'lucky'/correct wire is Y.

(a) Find P($\overline{B}$)

(b) Find the probability that the lucky wire is red given B.

(c) Find the probability that the lucky wire is yellow given B.

(d) Which wire should you cut?

Solution:

a) 3 Part LTP. We partition on the three cases for which wire is 'lucky'. Each of $P(R), P(Y), P(B) = 1/3$ assuming the bomber chose the wire randomly and uniformly. Then $P(\overline{B}|R) = 1/2$ (we guessed R, conditioning on R being the lucky wire, the bomber chooses randomly from the remaining two) $P(\overline{B}|Y)1$ (only has a single non-lucky choice). $P(\overline{B}|B) = 0$ (the bomber only reveals non lucky wires). The sum of the LTP is $1/2$

b,c) I'll keep the explanation brief as this is just a reformulation of the classic Monty Hall problem. We ultimately want to find $P(R|\overline{B})$ and $P(Y|\overline{B})$. Using Bayes Theorem, we can rearrange/decompose this into probabilities that we have already computed. Answer: $P(R|\overline{B}) = 1/3$ $P(Y|\overline{B}) = 2/3$

Notice that we don't need to find P(B | B) since that is guaranteed to be 0.

d) Cut Yellow

Expert profiler Manav Rao, has informed you of one of the bombers 'tells'. When you pretend to cut the 'lucky' wire, the bomber scratches their chin $\frac{3}{5}$ of the time. When you pretend to cut a different wire, the bomber scratches their chin $\frac{1}{10}$ of the time.

Let $\overline{B}$ be the event that the bomber reveals B to be the incorrect choice and scratches their chin.

Let R be the event that the 'lucky'/correct wire is R.

Let Y be the even that the 'lucky'/correct wire is Y.

(a) Find the probability that the lucky wire is red given B.

(b) Find the probability that the lucky wire is yellow given B.

(c) Which wire should you cut?

Solution:

a,b) Steps are the same as the previous one with the added condition. Finding $P(\overline{B})$ will remain the same; use LTP, partitioning off of R,Y,B. The numbers in this part will be slightly different than the previous. LTP sum = 2/15. Then we can compute the two conditional probabilities the same way. $P(R|\overline{B}) = 3/4$ $P(Y|\overline{B}) = 1/4$

c) Cut R (don't switch after the information is revealed).